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INEQUALITIES FOR THE CAPACITY OF A CONDENSER.*

By G. Pólya and G. Szegö.

1. Introduction. 1. Let us consider a solid S, a plane ϵ, a straight line e, and a point E. The symmetrization of S with respect to the plane ϵ is defined as a geometrical transformation changing the solid S into a solid S' symmetrical with respect to ϵ and characterized by the following properties. Any straight line perpendicular to ϵ and intersecting one of the solids S and S' intersects also the other and both intersections have the same length; the intersection with S' is a segment unless it is empty or it degenerates into a point.

Symmetrization with respect to the line e changes S into a solid of revolution S'' with axis e defined by the following properties. Any plane perpendicular to e and intersecting one of the solids S and S'' intersects also the other and both intersections have the same area; the intersection with S'' is a circular disk unless it is empty or it degenerates into a point.

Symmetrization with respect to the point E changes the solid S into a sphere S''' with center at E; S and S''' have the same volume.

The first kind of symmetrization, with respect to a plane, may be called the Steiner symmetrization; the second kind, with respect to a straight line, the Schwarz symmetrization. The Steiner symmetrization repeated for a suitably chosen sequence of planes generates the other two kinds of symmetrization.

Condensing three statements into one we may say that *all three kinds of symmetrization preserve the volume and diminish* (*or leave unchanged*) *the surface of the solid.* That part of this condensed statement which is concerned with the symmetrization with respect to a point is the classical isoperimetric property of the sphere at the proof of which J. Steiner and H. A. Schwarz aimed in introducing the geometrical transformations described above. [See for instance T. Bonnesen-W. Fenchel **2**, pp. 69-72].[1]

2. Let us consider two closed surfaces σ_0 and σ_1 such that σ_0 is contained in the interior of σ_1. We say that σ_0 and σ_1 form a "condenser" and we refer

* Received January 31, 1944.

[1] Bold-face figures refer to the Bibliography at the end of this paper; a reference such as **1. 2** is to Section **1**, part 2.

to the closed set of points exterior to σ_0 and interior to σ_1 as the "field." Let us consider a function ψ with continuous derivatives defined in the field and such that $\psi = 0$ on σ_0 and $\psi = 1$ on σ_1. We write

$$W(\psi) = (4\pi)^{-1} \int\int\int | \operatorname{grad} \psi |^2 \, dxdydz \tag{1.1}$$

the integration being extended over the field. The minimum of $W(\psi)$ obtained by varying ψ is the *capacity* of the condenser. (The minimum is attained when, and only when, ψ is harmonic.) The capacity of the solid bounded by σ_0 is defined as the capacity in the limiting case when σ_1 becomes an infinitely large sphere.

For other equivalent definitions of the capacity see G. Pólya and G. Szegö **15**, G. Szegö **20** and **21**.

3. We symmetrize the condenser by symmetrizing the two solids bounded by σ_0 and σ_1 simultaneously, with respect to the same plane or line or point. The present investigation is concerned mainly with the following statement: *All three kinds of symmetrization diminish* (*or leave unchanged*) *the capacity of the condenser.* That part of this compound statement which is concerned with the symmetrization with respect to a point is well known; it was discovered by H. Poincaré [**10**, pp. 17-22]. A different proof was suggested by G. Faber [**5**], and the first complete proof was given by Szegö [**20**]. The two other parts are new to our knowledge. For the proofs see Sections **2** and **3** below; the underlying intuitive idea is sketched in Section **2**. We treat all three cases of symmetrization by the same method which, in case of symmetrization with respect to a point, is essentially the same as that hinted at by Faber [**5**].

4. The corresponding definitions in the plane are rather obvious. We have in this case two kinds of symmetrization, one with respect to a straight line e and another with respect to a point E. The first kind, which was considered by Steiner, is in fact equivalent to symmetrization with respect to a plane perpendicular to the given plane and passing through e; the second changes any plane domain into a circle with the same area and with center at E.

We consider a closed curve s in the plane. Let us denote by A the area enclosed by s, by L the length of s, and by r_P the inner radius of s with respect to a point P situated in the interior of s; furthermore denote by $\bar{r}$ the outer radius of s, by C the capacity of the disk bounded by s, and by Λ the fundamental frequency of the membrane bounded by s. (For the definitions of r_P, $\bar{r}$, and Λ, see Section **4**).

Both kinds of symmetrization increase (or leave unchanged) r_P, preserve A, and diminish (or leave unchanged) L, $\bar{r}$, C, and Λ.

This statement is very much condensed. Let us consider first the symmetrization with respect to a line. Then the statement regarding A is trivial and that regarding L is classical; all the other statements are new to our knowledge. (In considering r_P we have to symmetrize not only s but also an infinitesimal circle including P. Thus, if we choose the line of symmetrization through P, P is not moved at all. There is an analogous remark about the next case, concerning the point of symmetrization.)

In case of symmetrization with respect to a point the statement about A is trivial and that concerning L is equivalent to the classical isoperimetric theorem. The statements about r_P and $\bar{r}$ are equivalent to the area theorems of Bieberbach [see Pólya and Szegö **14**, vol. 2, p. 21, problem 125, p. 22, problem 126]. The part concerned with Λ was suggested by Lord Rayleigh [**16**; **17**, p. 345] and proved by Faber [**5**]. The proof of the last author was rediscovered by E. Krahn [**7**, **8**] and preceded by a paper of R. Courant [**4**] who considered a kind of symmetrization different from that of Steiner. Finally the part concerning C is new so far as we know.

The following inequalities are equivalent to some of these statements (except the last one which is of a different nature, cf. for instance Pólya and Szegö **14**, vol. 2, p. 21, problem 124):

$$r_P \leqq (A/\pi)^{1/2} \leqq (\pi/2)C, \qquad (A/\pi)^{1/2} \leqq \bar{r} \leqq L/(2\pi). \tag{1.2}$$

These inequalities and various special cases suggest that

$$(\pi/2)C \leqq \bar{r} \tag{1.3}$$

which however we were not able to prove. We have shown in a former investigation [**15**] that $C \leqq \bar{r}$.

5. Returning again to the three-dimensional case we state the following further relation of the capacity C of a closed surface σ_0 to certain geometrical properties of σ_0 [J. C. Maxwell **9**, p. 117, Szegö **21**]:

Consider all closed surfaces σ_0 with given diameter. The capacity C of σ_0 is a maximum if and only if σ_0 is a sphere.

By the diameter of σ_0 we mean the greatest distance between any two of its points. Since the capacity is an increasing set-function it is sufficient to prove this theorem for *convex* surfaces. While the result of Poincaré-Faber-

Szegö gives a lower bound for C the theorem mentioned furnishes an upper bound for the same quantity.

In another part of the present paper we prove the following refinement of this theorem:

Let the inner surface σ_0 of a condenser of capacity C be convex and the outer surface σ_1 be parallel to σ_0. Let h be the distance between σ_0 and σ_1, A the area of the surface σ_0 and M the integral of the mean curvature of σ_0. Then

$$A/(4\pi h) < C < A/(4\pi h) + M/(4\pi). \tag{1.4}$$

For concentric spheres the second inequality becomes an equality.

It is customary to regard the left-hand side of (1.4) as an approximation to C for small h; see for instance Maxwell **9**, p. 140. On the other hand R. Clausius [**3**, p. 43] suggested an approximate calculation of the electric density according to which the right-hand side of (1.4) would furnish *a better approximation than the left-hand side.*

Concerning the limiting case $h \to \infty$ see **21**, p. 425.

We also prove the analogue of the inequalities (1.4) for the logarithmic potential.

6. The remaining part of this paper is devoted to the study of the capacity in interesting special cases, especially in that of an *ellipsoid.* Let us consider first the capacity C (see **1.2** above) of any convex solid S. Let V denote the volume of S, A the area of the surface of S, and M the integral of the mean curvature extended over the surface of S. It is well known [**2**, p. 110] that

$$(3V/(4\pi))^{1/3} < (A/(4\pi))^{1/2} < M/(4\pi) \tag{1.5}$$

unless S is a sphere. As a limiting case of the inequalities which we discussed in **1.3** and **1.5** we have

$$(3V/(4\pi))^{1/3} < C < M/(4\pi) \tag{1.6}$$

unless S is a sphere. For the special case of an ellipsoid with semi-axes a, b, c we prove the following refinement of (1.6):

(1.7)

$$(3V/(4\pi))^{1/3} < \tfrac{1}{3}\{(bc)^{1/2} + (ca)^{1/2} + (ab)^{1/2}\} < C < \tfrac{1}{3}(a + b + c) < M/(4\pi)$$

unless $a = b = c$, that is, unless the ellipsoid is a sphere. The proofs are

based on some representations of the capacity of an ellipsoid, related to a representation of its area, which played a role in a recent investigation of the first author [11].

7. The quantity $(A/(4\pi))^{1/2} = R_A$ occurring in (1.5) is obviously the radius of the sphere whose surface is equal to that of the given solid. In connection with the inequalities (1.5) and (1.6) we observe that the capacity C is "not comparable" with R_A, that is, for certain solids C is greater and for others smaller than R_A. A. Russell [19], following a hint of Lord Kelvin, gives a numerical discussion of the ratio of these quantities in various interesting cases especially for ellipsoids of revolution (spheroids). Let β be the numerical excentricity of the revolving ellipse; then Russell proves that the expansion of the ratio C/R_A in a power series in β begins thus:

$$C/R_A = 1 \pm (2/945)\beta^6 + \cdots \tag{1.8}$$

where the signs $\pm$ hold according to the case of a prolate and an oblate ellipsoid, respectively. [Cf. also the earlier papers of K. Aichi **1** and Lord Rayleigh **18**.]

We prove that for an ellipsoid of revolution

$$C/R_A \gtrless 1 \tag{1.9}$$

according as it is prolate or oblate. More precisely, we prove that the expansion of the quantity $(R_A/C)^2$ in a power series in β^2 has only negative and positive coefficients, respectively, except for the constant term 1 in the first case and the two subsequent coefficients which are zero in both cases.

8. The capacity of an ellipsoid is expressed by an elliptic integral [(6.4)]. In particular the capacity of an elliptic disk is the arithmetic-geometric mean of the semi-axes a, b multiplied by $2/\pi$. Hence the two elliptic disks with semi-axes a, b and $(a+b)/2$, $(ab)^{1/2}$, respectively, have the same capacity. Using Landen's transformation we find a somewhat analogous transition from one ellipsoid to another preserving the capacity.

9. In Section **2** we prove that the capacity of a condenser is diminished by applying the Steiner symmetrization; in Section **3** we show the same for the Schwarz symmetrization. Section **4** is concerned with the two-dimensional case. In Section **5** we give the proof of the inequalities (1.4). Sections **6-9** deal with the special case of an ellipsoid, Section **10** with some further special cases.

For the sake of simplicity we assume that all surfaces and lines considered below are composed of a finite number of analytic pieces. It does not seem to be doubtful that this condition can be replaced by some more general one without altering the main line of our argument. Also our argument does not change if the solid bounded by σ_0 is multiply connected or consists of several separated pieces. This deserves to be emphasized since such topological differences may require essential modifications of the proof; see for instance Pólya 12.

2. Steiner symmetrization. In this section we prove that the capacity of a condenser is diminished by the Steiner symmetrization.

1. The main idea of the proof is very simple; it can be explained by using the notation of **1. 2**. We denote by σ'_0 and σ'_1 the surfaces arising from σ_0 and σ_1 by symmetrization. Let $\psi = u = u(x, y, z)$ be the (harmonic) function for which the minimum C of the integral (1. 1) is attained. We wish to construct a function $\psi' = v = v(x, y, z)$ defined in the field between σ'_0 and σ'_1, equal to 0 on σ'_0 and to 1 on σ'_1, and such that

$$W(u) \geqq W'(v). \tag{2.1}$$

Here W' denotes the expression analogous to (1. 1) for the symmetrized condenser; the integration in W' is extended over the field between σ'_0 and σ'_1.

If such a construction is possible the assertion is proved. Indeed the left-hand side of (2. 1) is equal to C while the right-hand side is not less than the capacity C' of the symmetrized condenser.

As to the construction of v we consider the level surfaces σ_λ of u defined by the equation

$$u = \lambda; \qquad 0 \leqq \lambda \leqq 1. \tag{2.2}$$

Since u is harmonic we have $0 \leqq u \leqq 1$ in the field between σ_0 and σ_1. Thus the level surfaces (2. 2) are all in this field and exhaust it completely; that is, through each point of the field passes a definite surface (2. 2). Observe that σ_β contains σ_α in its interior if $\beta > \alpha$. Let σ'_λ be the surface arising from σ_λ by the symmetrization in question. Through each point of the field between σ'_0 and σ'_1 passes a definite surface σ'_λ; an easy geometric consideration confirms that σ'_β contains σ'_α if σ_β contains σ_α. Now we define $\psi' = v$ by "transplanting the parameter," that is, we ascribe to $\psi' = v$ the value λ at each point of the surface σ'_λ.

The remaining part of the proof is devoted to the verification of (2. 1).

2. We assume that the plane ϵ with respect to which we symmetrize coincides with the x, y-plane. Let

$$z = z_\mu(x, y, \lambda), \qquad (\mu = 1, 2, \cdots, 2m), \tag{2.3}$$

be the portions of the surface σ_λ in the neighborhood of the points at which the vertical line through $(x, y, 0)$ intersects σ_λ and suppose that $z_1 > z_2 > z_3 > \cdots > z_{2m}$. The number $2m = 2m(x, y)$ of these points is finite. We prove that the total contribution of these $2m$ points to the Dirichlet integral $W(u)$ is not less than the contribution of the corresponding two points of the symmetrized surface σ'_λ to the integral $W'(v)$.

Denoting by γ_μ the angle between the exterior normal vector of σ_λ at (x, y, z_μ) and the positive z-axis we have

$$\cos \gamma_\mu = \pm \{1 + (\partial z_\mu/\partial x)^2 + (\partial z_\mu/\partial y)^2\}^{-1/2} \tag{2.4}$$

with the $+$ or $-$ sign according as μ is odd or even. Now grad u is parallel to the exterior normal vector of σ_λ and $|\operatorname{grad} u| = \partial\lambda/\partial n$ so that

$$(\partial\lambda/\partial n) \cos \gamma_\mu = \partial\lambda/\partial z_\mu. \tag{2.5}$$

Here $(-1)^{\mu-1}\partial\lambda/\partial z_\mu > 0$. Hence the contribution of the μ-th point to $W(u)$ will be

$$\begin{aligned}(\partial\lambda/\partial n)^2 dx\, dy\, dz &= (\partial\lambda/\partial n)^2 dx\, dy\, |dz_\mu| \\ &= (\cos\gamma_\mu)^{-2}(\partial\lambda/\partial z_\mu)^2 dx\, dy\, |\partial z_\mu/\partial\lambda|\, d\lambda \\ &= \frac{1 + (\partial z_\mu/\partial x)^2 + (\partial z_\mu/\partial y)^2}{|\partial z_\mu/\partial\lambda|}\, dx\, dy . d\lambda,\end{aligned} \tag{2.6}$$

where $d\lambda > 0$, from which we conclude that the total contribution is

$$dx\, dy\, d\lambda \sum_{\mu=1}^{2m} \frac{1 + (\partial z_\mu/\partial x)^2 + (\partial z_\mu/\partial y)^2}{|\partial z_\mu/\partial\lambda|}. \tag{2.7}$$

The corresponding quantity for the symmetrized domain can be obtained from this result by substituting $m = 1$ and replacing z_1 and $-z_2$ by

$$\tfrac{1}{2}(z_1 - z_2 + z_3 - z_4 + \cdots + z_{2m-1} - z_{2m}). \tag{2.8}$$

We find

$$dx\, dy\, d\lambda \, \frac{4 + (\sum_{\mu=1}^{2m} (-1)^{\mu-1}\partial z_\mu/\partial x)^2 + (\sum_{\mu=1}^{2m} (-1)^{\mu-1}\partial z_\mu/\partial y)^2}{\sum_{\mu=1}^{2m} (-1)^{\mu-1}\partial z_\mu/\partial\lambda}. \tag{2.9}$$

Our assertion follows from the elementary inequality

$$(2.10)\qquad \sum_{\mu=1}^{M} \frac{1+p_\mu^2+q_\mu^2}{r_\mu} \geqq \frac{4+(\sum_{\mu=1}^{M} p_\mu)^2+(\sum_{\mu=1}^{M} q_\mu)^2}{\sum_{\mu=1}^{M} r_\mu}$$

where p_μ, q_μ, r_μ are arbitrary real numbers, $r_\mu > 0$, $M \geqq 2$. This inequality is a simple consequence of the inequalities of Cauchy and Minkowski since

$$(2.11)\qquad \sum_{\mu=1}^{M} \frac{1+p_\mu^2+q_\mu^2}{r_\mu} \sum_{\mu=1}^{M} r_\mu \geqq \{\sum_{\mu=1}^{M} (1+p_\mu^2+q_\mu^2)^{1/2}\}^2$$

and

$$(2.12)\qquad \sum_{\mu=1}^{M} (1+p_\mu^2+q_\mu^2)^{1/2} \geqq \{(\sum_{\mu=1}^{M} 1)^2+(\sum_{\mu=1}^{M} p_\mu)^2+(\sum_{\mu=1}^{M} q_\mu)^2\}^{1/2}.$$

In our case $M=2m \geqq 2$ so that $(\sum_{\mu=1}^{M} 1)^2 \geqq 4$. There is equality in (2.10) only for $M=2$, $p_1=p_2$, $q_1=q_2$, $r_1=r_2$.

3. Schwarz symmetrization. We now prove that the capacity of a condenser is diminished (or remains unchanged) by applying the process of the Schwarz symmetrization. Let the x-axis be the axis of the symmetrization. We retain the notations u, σ_λ introduced in Section **2** and define σ'_λ, v in the same way as in Section **2** replacing only the Steiner symmetrization by the Schwarz symmetrization.

1. We denote the area of the cross-section of σ_λ corresponding to a certain value of x by $A=A(x,\lambda)$. For the sake of simplicity we assume that this cross-section is bounded by a single closed curve which will be denoted by $s=s(x,\lambda)$.

We first compare $s(x,\lambda)$ and $s(x+dx,\lambda)$. Let $(\eta u_y, \eta u_z)$ be the normal vector of $s(x,\lambda)$ at (x,y,z) the end-point of which is on the projection of $s(x+dx,\lambda)$ onto the plane of $s(x,\lambda)$. Then

$$u(x+dx, y+\eta u_y, z+\eta u_z)=\lambda,$$

$$(3.1)\qquad u_x dx+\eta(u_y^2+u_z^2)=0, \qquad \eta=-u_x(u_y^2+u_z^2)^{-1}dx,$$

so that

$$(3.2)\qquad |A_x| \leqq \int_{s(x,\lambda)} |u_x| (u_y^2+u_z^2)^{-1/2} ds$$

where ds denotes the arc element of $s(x,\lambda)$.

On the other hand let $(\eta' u_y, \eta' u_z)$ be the normal vector of $s(x, \lambda)$ at (x, y, z) with end-point on $s(x, \lambda + d\lambda)$, $d\lambda > 0$. Then

$$(3.3) \qquad u(x, y + \eta' u_y, z + \eta' u_z) = \lambda + d\lambda, \qquad \eta'(u_y^2 + u_z^2) = d\lambda$$

so that

$$(3.4) \qquad | A_\lambda | = \int_{s(x,\lambda)} (u_y^2 + u_z^2)^{-1/2} ds.$$

2. Let us consider the ring-shaped domain between σ_λ and $\sigma_{\lambda+d\lambda}$, $d\lambda > 0$, which is cut off by the two symmetrizing planes through $(x, 0, 0)$ and $(x + dx, 0, 0)$. We prove as in Section 2 that the contribution of this domain to $W(u)$ is not less than the contribution of the corresponding part of the symmetrized domain to $W'(v)$.

Let (x, y, z) be an arbitrary point of σ_λ, that is $u(x, y, z) = \lambda$. Then

$$(3.5) \qquad \partial z/\partial x = -u_x/u_z, \qquad \partial z/\partial y = -u_y/u_z, \qquad \partial z/\partial\lambda = 1/u_z$$

and so from (2.6) we obtain the contribution

$$(3.6) \qquad dx\, d\lambda \int \frac{1 + u_x^2/u_z^2 + u_y^2/u_z^2}{1/| u_z |} | dy |$$

the integration being extended over $s(x, \lambda)$. But if x and λ are fixed, $u_y dy + u_z dz = 0$; hence

$$(3.7) \qquad | dy | = | u_z | (u_y^2 + u_z^2)^{-1/2} ds$$

and (3.6) becomes

$$(3.8) \qquad dxd\lambda \int_{s(x,\lambda)} (u_x^2 + u_y^2 + u_z^2)(u_y^2 + u_z^2)^{-1/2} ds.$$

3. The geometric condition defining the symmetrized surface σ'_λ can be expressed in the form $A(x, \lambda) = \pi(y^2 + z^2)$ and this equation defines the symmetrized function $\lambda = v(x, y, z)$. The intersection of σ'_λ with the plane through $(x, 0, 0)$ perpendicular to the x-axis is a circle with area A. We find easily that

$$(3.9) \qquad v_x = -A_x/A_\lambda, \qquad v_y = 2\pi y/A_\lambda, \qquad v_z = 2\pi z/A_\lambda$$

and so the corresponding contribution to (3.8) will be

$$(3.10) \qquad dxd\lambda\{A_x^2 A_\lambda^{-2} + 4\pi^2(y^2 + z^2)A_\lambda^{-2}\}\{4\pi^2(y^2 + z^2)A_\lambda^{-2}\}^{-1/2} 2\pi(A/\pi)^{1/2}$$
$$= dxd\lambda(A_x^2 + 4\pi A)A_\lambda^{-1}.$$

Thus we have to prove that

$$(3.11) \qquad (A_x^2 + 4\pi A)A_\lambda^{-1} \leqq \int_{s(x,\lambda)} (u_x^2 + u_y^2 + u_z^2)(u_y^2 + u_z^2)^{-1/2} ds.$$

We use (3.2) and (3.4). By Schwarz's inequality

$$(3.12) \qquad A_x^2 \leqq A_\lambda \int_{s(x,\lambda)} u_x^2 (u_y^2 + u_z^2)^{-1/2} ds,$$

and by the isoperimetric property of the circle:

$$(3.13) \qquad 4\pi A \leqq \left(\int_{s(x,\lambda)} ds\right)^2 \leqq A_\lambda \int_{s(x,\lambda)} (u_y^2 + u_z^2)^{1/2} ds.$$

Hence

$$(3.14) \qquad A_x^2 + 4\pi A \leqq A_\lambda \int_{s(x,\lambda)} (u_x^2 + u_y^2 + u_z^2)(u_y^2 + u_z^2)^{-1/2} ds,$$

which agrees with the inequality (3.11).

4. It is instructive to compare the proofs given in Section **2** and in this section with the proof suggested by Faber [5] that the capacity is diminished or remains unchanged by the *third* kind of symmetrization defined in the Introduction. This symmetrization replaces the level surface σ_λ $(u = \lambda)$ by a sphere σ'_λ of equal volume (with a fixed center) and the "symmetrized" function v is defined as $v = \lambda$ on σ'_λ. Let us denote by $V = V(\lambda)$ and $R = R(\lambda)$ the volume and the radius of this sphere, respectively.

We compare the contribution of the shell between σ_λ and $\sigma_{\lambda+d\lambda}$, $d\lambda > 0$, to the integral $W(u)$ with the corresponding contribution of the spherical shell between σ'_λ and $\sigma'_{\lambda+d\lambda}$ to $W'(v)$. Since the piece of the normal of the surface σ_λ between σ_λ and $\sigma_{\lambda+d\lambda}$ is $|\operatorname{grad} u|^{-1} d\lambda$ we find for the corresponding part of $W(u)$

$$(3.15) \qquad \int\int |\operatorname{grad} u|^2 |\operatorname{grad} u|^{-1} d\lambda d\sigma = d\lambda \int_{\sigma_\lambda}\int |\operatorname{grad} u| \, d\sigma$$

where $d\sigma$ is the area element of σ_λ.

For the symmetrized function v we find

$$(3.16) \qquad |\operatorname{grad} v| = d\lambda/dR = \{R'(\lambda)\}^{-1} = 3\{R(\lambda)\}^{-1} V(\lambda) \{V'(\lambda)\}^{-1}$$

and so the corresponding contribution is

$$(3.17) \qquad d\lambda \cdot 3\{R(\lambda)\}^{-1} V(\lambda)\{V'(\lambda)\}^{-1} \cdot 4\pi\{R(\lambda)\}^2.$$

Thus we have to prove that

$$(3.18) \qquad \int_{\sigma_\lambda}\int |\operatorname{grad} u|\, d\sigma \geqq (36\pi)^{2/3}\{V(\lambda)\}^{4/3}\{V'(\lambda)\}^{-1}.$$

But by Schwarz's inequality and by the isoperimetric property of the sphere

$$(3.19) \qquad \int_{\sigma_\lambda}\int |\operatorname{grad} u|\, d\sigma \int_{\sigma_\lambda}\int |\operatorname{grad} u|^{-1} d\sigma \geqq \Big(\int_{\sigma_\lambda}\int d\sigma\Big)^2 \geqq (36\pi)^{2/3}\{V(\lambda)\}^{4/3}.$$

The second integral on the left-hand side is $V'(\lambda)$. This proves the assertion.

4. The two-dimensional case. In this section we deal with "two-dimensional condensers" bounded by closed curves s_0 and s_1, s_0 interior to s_1. We shall be concerned in particular with the statements formulated in **1.4** concerning r_P, $\bar{r}$, C, and Λ.

1. The "two-dimensional capacity" or, shorter, *capacity* of a two-dimensional condenser bounded by s_0 and s_1 can be defined as the minimum of

$$(4.1) \qquad (2\pi)^{-1}\int\int |\operatorname{grad} \psi|^2 dxdy$$

the integration being extended over the set of points between s_0 and s_1; here ψ is an arbitrary function with continuous derivatives defined in the domain of integration and satisfying the conditions: $\psi = 0$ on s_0, $\psi = 1$ on s_1. This capacity differs from the "three-dimensional" capacity of the ring-shaped disk bounded by s_0 and s_1. It is the double of the "capacity per unit height" of a condenser formed by two infinite cylinders; it is related to the concept of the logarithmic potential:

The capacity of a two-dimensional condenser diminishes (or remains unchanged) by symmetrization with respect to a straight line in the plane of the condenser.

The proof follows exactly the same line as in Section **2**.

2. Let s be a closed curve in a plane, P a fixed point in the interior of s. The *inner radius* r_P of s with respect to P can be defined as the radius of the uniquely determined circle around P such that the interior of s can be mapped conformally onto the interior of this circle preserving the point P and the line-element (i. e. the magnitude and direction of the line-element) at P.

If we introduce complex numbers $z = x + iy$ in the plane of s and complex numbers w in the plane of the circle and if P is the point $z = w = 0$ this mapping can be represented in the form

$$w = \phi(z) = z + c_2 z^2 + c_3 z^3 + \cdots, \qquad |w| < r_P, \tag{4.2}$$

where $\phi(z)$ is a power series in z.

Before we discuss the effect of the symmetrization on the inner radius r_P let us consider another formulation of the previous definition. We denote by $G = G(x, y)$ the Green's function of the interior of s with respect to the point $P(0, 0)$ defined by the following properties:

(i) G is harmonic in the interior of s except at P;

(ii) $G = 0$ on s;

(iii) $G = \log(1/r) + g$

where r is the distance from P and $g = g(x, y)$ is regular-harmonic in the interior of s. Then for the inner radius r_P of s with respect to P we obtain

$$\log r_P = g(0, 0). \tag{4.3}$$

Indeed, using the notation (4.2), we have

$$G(x, y) = \log(r_P / |\phi(z)|). \tag{4.4}$$

The definition (4.3) has the advantage that it can be extended without difficulty to the three-dimensional case (in which conformal mapping is not possible in general). See Szegö **20**, p. 587. We now prove the following theorem:

The inner radius r_P of the curve s with respect to an interior point P is not decreased by symmetrizing s with respect to a straight line in its plane.

A similar theorem holds in space (cf. Szegö **20**).

For the proof let us consider the level curve $G = \log(r_P/\eta)$ or $|\phi(z)| = \eta$ where $\eta > 0$, $\eta \to 0$. This level curve obviously lies in the circular ring

$$\eta - a\eta^2 < |z| < \eta + a\eta^2 \tag{4.5}$$

where a is a certain positive constant independent of η. Now let the constant ω be fixed, $\omega > a$. If δ is a sufficiently small positive number and

$$\eta' = \delta - \omega\delta^2, \qquad \eta'' = \delta + \omega\delta^2 \tag{4.6}$$

we have

$$\eta' + a\eta'^2 < \delta < \eta'' - a\eta''^2. \tag{4.7}$$

This means that the circle $|z| = \delta$ is between the two level curves $G = \log(r_P/\eta')$ and $G = \log(r_P/\eta'')$.

But the two-dimensional condenser bounded by the curve s and the level curve $G = \log(r_P/\eta)$ has the capacity [2]

$$\{\log(r_P/\eta)\}^{-1}. \tag{4.8}$$

Hence the condenser bounded by s and the circle $|z| = \delta$ has a capacity which is between the quantities $\{\log(r_P/\eta')\}^{-1}$ and $\{\log(r_P/\eta'')\}^{-1}$ and so has the form

$$\{\log(r_P/\delta) + O(\delta)\}^{-1} \tag{4.9}$$

where $\delta^{-1}O(\delta)$ is bounded as $\delta \to 0$.

After these preliminaries we symmetrize the curve s with respect to a straight line through P. If the resulting curve s' has the inner radius r'_P with respect to the same point P we find, since the circle $|z| = \delta$ is not changed by the symmetrization in question, that

$$\{\log(r_P/\delta) + O(\delta)\}^{-1} \geqq \{\log(r'_P/\delta) + O(\delta)\}^{-1} \tag{4.10}$$

and so $r_P \leqq r'_P$ which is the assertion.

3. The *outer radius* $\bar{r}$ of the curve s can be defined as the radius of the uniquely determined circle onto the exterior of which the exterior of s can be mapped conformally so that the point at infinity and the line element in it are preserved.

If we introduce complex numbers $z = x + iy$ in the plane of s and complex numbers w in the plane of the circle this mapping can be represented in the form

$$w = \chi(z) = z + c_0 + c_1 z^{-1} + c_2 z^{-2} + \cdots, \qquad |w| > \bar{r}. \tag{4.11}$$

A definition of $\bar{r}$ in terms of Green's function is again possible. Moreover the outer radius is identical with the *transfinite diameter* of s **[15]**.

Let us consider a condenser with s as interior curve and with a large circle of radius R as exterior curve. Then the capacity will be approximately [cf. **15**, p. 8]

[2] This is $(2\pi)^{-1}$ times the flux integral of the harmonic function $\{\log(r_P/\eta)\}^{-1}G$ extended over an arbitrary closed curve inside s and containing the level curve $G = \log(r_P/\eta)$.

$$\{\log(R/\bar{r})\}^{-1}. \tag{4.12}$$

From this it follows as above that *the outer radius of the curve s decreases (or remains unchanged) by symmetrizing s with respect to a straight line in the plane of s.*

4. We consider a cylindrical solid the base of which is bounded by a curve s; the generators may or may not be perpendicular to the bases. Applying the Schwarz symmetrization with an axis perpendicular to the bases of the solid we obtain the following remarkable theorem:

The capacity of a cylindrical solid is never less than the capacity of a right circular cylinder whose base has the same area as that of the given solid.

(The right circular cone has an analogous extremal property.)

In the limiting case in which the height of the cylinder becomes 0 we obtain the statement of **1.** 4 concerning the capacity of a disk. Since the capacity of a circular disk with radius 1 is $2/\pi$ [see (8.10)] we obtain the second inequality in (1.2).

5. Finally let us consider again the integral (4.1) extended over the interior of a given curve s. The minimum of this integral under the boundary condition $\psi = 0$ on s and the additional condition

$$\iint \psi^2 dxdy = 1 \tag{4.13}$$

furnishes a constant which can be called the *fundamental frequency* Λ of the disk bounded by s. In discussing the effect of symmetrization on the fundamental frequency of a disk the same argument can be followed as in the case of the capacity. If the original function u satisfies the condition $u = 0$ on s the same will hold for the "symmetrized function" v on the symmetrized curve s'. The normalizing condition (4.13) will also be preserved. This yields the assertion announced in **1.** 4 concerning Λ.

The analogous theorem in space is also true.

5. Proof of the inequalities (1.4). 1. Let σ_0 be a closed convex surface, σ_1 the outer parallel surface at the distance h, $h > 0$. We denote by $C_h = C$ the capacity of the condenser determined by σ_0 and σ_1. First we establish the lower bound for C given in (1.4).

This can be easily done by means of the minimum property of the integral (1.1). In order to perform the integration (1.1) over the field in the present

case we first integrate along the normals of σ_0 and then over σ_0. Let us denote by $\partial\psi/\partial k$ the derivative of ψ in the direction of a fixed normal of σ_0, $0 \leqq k \leqq h$. Then

$$\int_0^h |\operatorname{grad}\psi|^2\,dk \geqq \int_0^h (\partial\psi/\partial k)^2\,dk \tag{5.1}$$

and so, by Schwarz's inequality,

$$1 = \{\int_0^h (\partial\psi/\partial k)\,dk\}^2 \leqq h\int_0^h (\partial\psi/\partial k)^2\,dk; \tag{5.2}$$

hence

$$\int_0^h |\operatorname{grad}\psi|^2\,dk \geqq h^{-1}. \tag{5.3}$$

This yields the lower bound in (1.4).

2. The proof of the upper bound for C given in (1.4) requires a deeper analysis. If $u(h;x,y,z) = u(h;P)$ is the function harmonic in the field between σ_0 and σ_1 for which $u = 0$ on σ_0 and $u = 1$ on σ_1, we have

$$4\pi C_h = \iint (\partial u/\partial n)\,d\sigma \tag{5.4}$$

the integration being extended over an arbitrary closed surface in the field containing σ_0 (n the exterior normal). We show first that C_h, as a function of h, possesses a left-hand derivative $(d/dh)C_h$ and that

$$(d/dh)C_h = -(4\pi)^{-1}\int_{\sigma_1}\!\!\int (\partial u/\partial n)^2 d\sigma. \tag{5.5}$$

3. Let $g = g(x,y,z) = g(P)$ be the function harmonic in the field satisfying the conditions

$$\begin{cases} g = 0 & \text{on } \sigma_0, \\ g = \partial u/\partial n & (n \text{ the interior normal}) \text{ on } \sigma_1. \end{cases} \tag{5.6}$$

The flux-integral

$$4\pi\gamma = \iint (\partial g/\partial n)\,d\sigma \tag{5.7}$$

extended over a closed surface in the field surrounding σ_0 (n the exterior normal) is then independent of the choice of this surface.

We introduce the function

$$\begin{aligned} \bar{u}(x,y,z) &= \bar{u}(P) \\ &= u(h;P) - \epsilon g(P) + C_h^{-1}\gamma\epsilon\{u(h;P) - 1\} - (C_h/C_{h-\epsilon})u(h-\epsilon;P) \end{aligned} \tag{5.8}$$

where $\epsilon > 0$. Let $P(x, y, z)$ be an arbitrary point on σ_1 and let PQ be an interior normal-vector of σ_1 of length ϵ so that Q is on the parallel surface σ'_1 at distance $h - \epsilon$ from σ_0. Then at Q we have

$$
\begin{aligned}
(5.9)\quad \bar{u}(Q) &= u(h; Q) - \epsilon g(Q) + C_h^{-1}\gamma\epsilon\{u(h; Q) - 1\} - C_h/C_{h-\epsilon} \\
&= u(h; P) + \epsilon(\partial u/\partial n)_P - \epsilon g(Q) + C_h^{-1}\gamma\epsilon\{u(h; P) - 1\} \\
&\quad - C_h/C_{h-\epsilon} + O(\epsilon^2) = 1 - C_h/C_{h-\epsilon} + O(\epsilon^2)
\end{aligned}
$$

(n the interior normal). On the other hand if (x, y, z) is on σ_0 then

$$(5.10)\qquad \bar{u} = - C_h^{-1}\gamma\epsilon.$$

Finally for the flux-integral we have

$$(5.11)\qquad \iint (\partial \bar{u}/\partial n)\, d\sigma = 4\pi C_h - 4\pi\gamma\epsilon + C_h^{-1}\gamma\epsilon \cdot 4\pi C_h - 4\pi C_h = 0.$$

But such a function $\bar{u}$ must assume on σ'_1 its maximum and minimum in the ring-shaped domain between σ_0 and σ'_1. Indeed σ_0 is a level surface. Hence if $\bar{u}$ would attain its maximum (minimum) on σ_0 we should have $\partial \bar{u}/\partial n \leqq 0$ ($\geqq 0$) (n the exterior normal). This together with (5.11) would imply that $\partial \bar{u}/\partial n = 0$ which is impossible.

Hence

$$(5.12)\qquad 1 - C_h/C_{h-\epsilon} - a\epsilon^2 < - C_h^{-1}\gamma\epsilon < 1 - C_h/C_{h-\epsilon} + a\epsilon^2$$

where a is a positive constant independent of ϵ. From this we conclude that

$$(5.13)\qquad \lim_{\epsilon \to +0} \frac{C_h - C_{h-\epsilon}}{\epsilon} = \gamma.$$

It is possible to express γ in terms of u; for this purpose we apply Green's formula:

$$
\begin{aligned}
(5.14)\quad &\int_{\sigma_0}\!\!\int (g(\partial u/\partial n) - u(\partial g/\partial n))\, d\sigma \\
&\qquad + \int_{\sigma_1}\!\!\int (g(\partial u/\partial n) - u(\partial g/\partial n))\, d\sigma = 0
\end{aligned}
$$

where the normal is directed into the interior of the field. The first integral vanishes and so

$$(5.15)\qquad \int_{\sigma_1}\!\!\int (\partial u/\partial n)^2 d\sigma = \int_{\sigma_1}\!\!\int (\partial g/\partial n)\, d\sigma = - 4\pi\gamma.$$

Combining (5.15) with (5.13) we obtain the fundamental formula (5.5).

4. Let A_h denote the area of the surface σ_1. By using Schwarz's inequality we conclude from (5. 4) and (5. 5) that

$$(5.16) \quad (4\pi C_h)^2 \leqq \int_{\sigma_1}\int (\partial u/\partial n)^2 d\sigma \int_{\sigma_1}\int d\sigma = -4\pi A_h (d/dh) C_h$$

or

$$(5.17) \quad (d/dh)(C_h^{-1}) \geqq 4\pi A_h^{-1}.$$

Integrating we obtain, since $\lim C_h^{-1} = 0$ as $h \to +0$,

$$(5.18) \quad C_h^{-1} \geqq 4\pi \int_0^h A_k^{-1} dk.$$

In the limiting case where σ_1 is a sphere with infinitely large radius this inequality was obtained by Szegö [**21**, p. 423]. The method used in the present proof is very similar to that used in the special case mentioned.

5. Let A and M have the same meanings as in (1. 4). It is well known [cf. for instance Szegö **21**, p. 424, (11), (12)] that

$$(5.19) \quad A_k = A + 2Mk + 4\pi k^2 \leqq A^{-1}(A + Mk)^2;$$

therefore

$$(5.20) \quad C_h^{-1} \geqq 4\pi A \int_0^h (A + Mk)^{-2} dk = 4\pi h (A + Mh)^{-1}$$

and this establishes the upper estimate of C given in (1. 4).

6. Also we have, more precisely,

$$(5.21) \quad C_h^{-1} \geqq 4\pi \int_0^h (A + 2Mk + 4\pi k^2)^{-1} dk.$$

Performing the integration

$$(5.22) \quad C_h^{-1} \geqq \frac{2\pi}{\lambda M} \log\left(\frac{(1-\lambda)M + 4\pi h}{(1+\lambda)M + 4\pi h} \cdot \frac{1+\lambda}{1-\lambda}\right); \quad \lambda = (1 - 4\pi A M^{-2})^{1/2}.$$

7. Finally we point out the inequalities corresponding to (1. 4) in the two-dimensional case where the capacity is associated with the theory of the logarithmic potential.

Let s_0 be a closed *convex* curve in the plane and s_1 an arbitrary outer curve parallel to s_0. We denote by C the capacity (defined in **4.** 1) of the two-dimensional condenser bounded by s_0 and s_1. If L_0 is the length of s_0 and L_1 the length of s_1 we have

$$(L_1/L_0 - 1)^{-1} < C < (\log(L_1/L_0))^{-1} \tag{5.23}$$

unless s_0 is a circle.

The proof follows exactly the same line of argument as in the case of the Newtonian potential.

6. Capacity of an ellipsoid. 1. Let a, b, c be the semi-axes of an ellipsoid and suppose that

$$a \geqq b \geqq c > 0. \tag{6.1}$$

Let α, β, γ be the numerical excentricities of the principal sections, each section being perpendicular to the respective axis:

$$\alpha^2 = 1 - c^2/b^2, \quad \beta^2 = 1 - c^2/a^2, \quad \gamma^2 = 1 - b^2/a^2. \tag{6.2}$$

We denote the capacity of this ellipsoid by $C = C(a, b, c)$.

The following representations of C are well known [cf. for instance E. Heine **6**, p. 156]:

$$C^{-1} = (4\pi)^{-1} \iint (a^2\xi^2 + b^2\eta^2 + c^2\zeta^2)^{-\frac{1}{2}} d\omega \tag{6.3}$$

$$= (1/2) \int_0^\infty \{(a^2 + u)(b^2 + u)(c^2 + u)\}^{-\frac{1}{2}} du. \tag{6.4}$$

The first integral is extended over the unit sphere $\xi^2 + \eta^2 + \zeta^2 = 1$ with the surface element $d\omega$.

2. Two further representations of C^{-1} in the form of infinite series can easily be obtained from (6.4). Straight-forward calculation yields

$$C^{-1} = (2a)^{-1} \int_0^\infty \{(1 + u)(1 - \beta^2 + u)(1 - \gamma^2 + u)\}^{-\frac{1}{2}} du$$

$$= (2a)^{-1} \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} g_\mu g_\nu \beta^{2\mu} \gamma^{2\nu} \int_0^\infty (1 + u)^{-3/2-\mu-\nu} du \tag{6.5}$$

where we use the abbreviation

$$g_0 = 1; \quad g_m = \frac{1 \cdot 3 \cdots (2m - 1)}{2 \cdot 4 \cdots 2m}, \qquad (m = 1, 2, 3, \cdots). \tag{6.6}$$

Thus we have the following formula:

$$C^{-1} = a^{-1} \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{g_\mu g_\nu \beta^{2\mu} \gamma^{2\nu}}{2(\mu + \nu) + 1} \tag{6.7}$$

$$= a^{-1} \sum_{n=0}^{\infty} \frac{(\beta\gamma)^n}{2n+1} P_n\left(\frac{\beta^2+\gamma^2}{2\beta\gamma}\right) \tag{6.8}$$

where $P_n(t)$ is Legendre's polynomial.

The second formula results from

$$C^{-1} = (2ab)^{-1}c \int_0^\infty \{(1+u)(1+u-\alpha^2 u)(1+u-\beta^2 u)\}^{-1/2} du$$

$$= (2ab)^{-1}c \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} g_\mu g_\nu \alpha^{2\mu}\beta^{2\nu} \int_0^\infty (1+u)^{-3/2-\mu-\nu} u^{\mu+\nu} du$$

$$= (ab)^{-1}c \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{g_\mu g_\nu}{g_{\mu+\nu}} \frac{\alpha^{2\mu}\beta^{2\nu}}{2(\mu+\nu)+1} \tag{6.9}$$

$$= (ab)^{-1}c \sum_{n=0}^{\infty} \frac{(\alpha\beta)^n}{(2n+1)g_n} P_n\left(\frac{\alpha^2+\beta^2}{2\alpha\beta}\right). \tag{6.10}$$

Here g_n has the same meaning as in (6.6).

7. Continuation. The following inequalities are known for the capacity of an ellipsoid [Pólya, Szegö **11**; cf. also Szegö **22** and Watson **23**]:

$$\frac{3}{a/(bc)+b/(ca)+c/(ab)} \leqq (abc)^{1/3} \leqq C \leqq \left(\frac{a^2+b^2+c^2}{3}\right)^{1/2} \leqq \frac{bc/a+ca/b+ab/c}{3}. \tag{7.1}$$

We observe that the inequality $(abc)^{1/3} \leqq C$ is equivalent to the special case of the theorem of Poincaré-Faber-Szegö mentioned in **1.** 3 in which σ_0 is an ellipsoid and σ_1 a large sphere.

Here we prove the more precise inequalities (1.7).

1. We begin with a proof for the upper bound of C. Taking (6.8) into account we can write the inequality in the following form:

$$\sum_{n=0}^{\infty} \frac{(\beta\gamma)^n}{2n+1} P_n\left(\frac{\beta^2+\gamma^2}{2\beta\gamma}\right) \geqq 3\{1+(1-\beta^2)^{1/2}+(1-\gamma^2)^{1/2}\}^{-1}. \tag{7.2}$$

This is equivalent to the inequality:

$$\sum_{n=0}^{\infty} \frac{h^n}{2n+1} P_n(t) \geqq 3(1+A^{1/2}+B^{1/2})^{-1} \tag{7.3}$$

where

$$\begin{cases} t>1, & 0<h<t-(t^2-1)^{1/2} \\ A=1-h\{t+(t^2-1)^{1/2}\}, & B=1-h\{t-(t^2-1)^{1/2}\}. \end{cases} \tag{7.4}$$

Instead of (7. 3) we prove the following stronger result:

$$(7.5)\qquad \frac{\partial}{\partial h}\sum_{n=0}^{\infty}\frac{h^{n+\frac{1}{2}}}{2n+1}P_n(t) \geqq \frac{\partial}{\partial h}\{3h^{\frac{1}{2}}(1+A^{\frac{1}{2}}+B^{\frac{1}{2}})^{-1}\}$$

or

$$(7.6)\qquad \sum_{n=0}^{\infty}h^nP_n(t) \geqq -3h(1+A^{\frac{1}{2}}+B^{\frac{1}{2}})^{-2}(A^{-\frac{1}{2}}\partial A/\partial h+B^{-\frac{1}{2}}\partial B/\partial h) + 3(1+A^{\frac{1}{2}}+B^{\frac{1}{2}})^{-1}.$$

The left-hand side is

$$(7.7)\qquad (1-2ht+h^2)^{-\frac{1}{2}} = (AB)^{-\frac{1}{2}}.$$

Since $h\partial A/\partial h = A - 1$, $h\partial B/\partial h = B - 1$ the inequality (7. 6) may be written as follows:

$$(7.8)\qquad (AB)^{-\frac{1}{2}} \geqq 3(1+A^{-\frac{1}{2}}+B^{-\frac{1}{2}})(1+A^{\frac{1}{2}}+B^{\frac{1}{2}})^{-2}$$

or

$$(7.9)\qquad (1+A^{\frac{1}{2}}+B^{\frac{1}{2}})^2 \geqq 3(A^{\frac{1}{2}}B^{\frac{1}{2}}+A^{\frac{1}{2}}+B^{\frac{1}{2}}).$$

This is equivalent to

$$(7.10)\qquad (A^{\frac{1}{2}}-B^{\frac{1}{2}})^2 \geqq -(1-A^{\frac{1}{2}})(1-B^{\frac{1}{2}}).$$

2. The proof for the lower bound follows a similar line but the details are more complicated. We prove that

$$(7.11)\qquad \sum_{n=0}^{\infty}\frac{(\beta\gamma)^n}{2n+1}P_n\left(\frac{\beta^2+\gamma^2}{2\beta\gamma}\right) \leqq 3\{(1-\beta^2)^{\frac{1}{4}}(1-\gamma^2)^{\frac{1}{4}}+(1-\beta^2)^{\frac{1}{4}}+(1-\gamma^2)^{\frac{1}{4}}\}^{-1}$$

or, in the previous notation,

$$(7.12)\qquad \sum_{n=0}^{\infty}\frac{h^n}{2n+1}P_n(t) \leqq 3(A^{\frac{1}{4}}B^{\frac{1}{4}}+A^{\frac{1}{4}}+B^{\frac{1}{4}})^{-1}.$$

Here again we prove more, namely the inequality

$$(7.13)\qquad \frac{\partial}{\partial h}\sum_{n=0}^{\infty}\frac{h^{n+\frac{1}{2}}}{2n+1}P_n(t) \leqq \frac{\partial}{\partial h}\{3h^{\frac{1}{2}}(A^{\frac{1}{4}}B^{\frac{1}{4}}+A^{\frac{1}{4}}+B^{\frac{1}{4}})^{-1}\}.$$

Using the previous calculations we can write (7. 13) in the form

$$\begin{aligned}(AB)^{-\frac{1}{2}} \leqq\ & 3(A^{\frac{1}{4}}B^{\frac{1}{4}}+A^{\frac{1}{4}}+B^{\frac{1}{4}})^{-1}\\ & -\tfrac{3}{2}(A^{\frac{1}{4}}B^{\frac{1}{4}}+A^{\frac{1}{4}}+B^{\frac{1}{4}})^{-2}\\ & \times\{A^{-\frac{3}{4}}B^{\frac{1}{4}}(A-1)+A^{\frac{1}{4}}B^{-\frac{3}{4}}(B-1)\\ & \qquad + A^{-\frac{3}{4}}(A-1)+B^{-\frac{3}{4}}(B-1)\},\end{aligned}$$

or

$$\begin{aligned}(7.14)\qquad 2(AB)^{1/4}(A^{1/4}B^{1/4}+A^{1/4}+B^{1/4})^2 &\leqq 6(AB)^{3/4}(A^{1/4}B^{1/4}+A^{1/4}+B^{1/4})\\ &\quad -3(AB)^{3/4}\{A^{-3/4}B^{1/4}(A-1)+A^{1/4}B^{-3/4}(B-1)\\ &\quad +A^{-3/4}(A-1)+B^{-3/4}(B-1)\}.\end{aligned}$$

Letting $A^{1/4}=r$, $B^{1/4}=s$, this inequality is equivalent to the statement that the polynomial of the seventh degree in r and s

$$(7.15)\qquad f(r,s)=3(r^4s^3+r^3s^4+r^4+s^4+r^3+s^3)-2rs(rs+r+s)^2$$

is *non-negative* for $0\leqq r\leqq 1$, $0\leqq s\leqq 1$. We give two proofs of this assertion.

3. *First proof.* We have

$$\begin{aligned}(7.16)\qquad f(r,s) &= 2(r^2-s^2)(r-s)+3(r^3-s^3)(r-s)\\ &\quad +rs(r-s)^2+2rs(r+s)(1-rs)^2\\ &\quad +2(\tfrac{3}{8}r^3+\tfrac{3}{8}s^3+\tfrac{1}{8}r^4s^3+\tfrac{1}{8}r^3s^4-r^2s^2)\\ &\quad +2(\tfrac{1}{8}r^3+\tfrac{1}{8}s^3+\tfrac{3}{8}r^4s^3+\tfrac{3}{8}r^3s^4-r^3s^3).\end{aligned}$$

The expressions in the last two rows are non-negative since

$$(7.17)\qquad (r^3)^{3/8}(s^3)^{3/8}(r^4s^3)^{1/8}(r^3s^4)^{1/8}=r^2s^2,\quad (r^3)^{1/8}(s^3)^{1/8}(r^4s^3)^{3/8}(r^3s^4)^{3/8}=r^3s^3.$$

Second proof. We show that $f(r,s)$ attains its minimum in the square for $r=s=0$ and for $r=s=1$, this minimum being 0.

Since for $0\leqq r\leqq 1$

$$\begin{aligned}(7.18)\qquad f(r,r) &= 6(r^7+r^4+r^3)-2r^4(r+2)^2\\ &= 2r^3(r-1)^2(3r^2+5r+3)\geqq 0,\\ f(r,0) &= 3(r^4+r^3)\geqq 0\end{aligned}$$

it is sufficient to show that $f(r,s)$ can not attain a relative minimum at an interior point of the square not on the diagonal $r=s$. Now

$$\begin{aligned}(7.19)\qquad f_r(r,s) &= 12r^3s^3+9r^2s^4+12r^3+9r^2\\ &\quad -2s(rs+r+s)^2-4rs(rs+r+s)(1+s),\\ f_s(r,s) &= 12r^3s^3+9s^2r^4+12s^3+9s^2\\ &\quad -2r(rs+r+s)^2-4rs(rs+r+s)(1+r)\end{aligned}$$

and so

$$\begin{aligned}(7.20)\qquad \frac{f_r(r,s)-f_s(r,s)}{r-s} &= -9r^2s^2(r+s)+12(r^2+rs+s^2)+9(r+s)\\ &\quad +2(rs+r+s)^2+4rs(rs+r+s)>0\end{aligned}$$

provided $0 \leqq r < s \leqq 1$ or $0 \leqq s < r \leqq 1$. For such values of r and s the two derivatives $f_r(r,s)$ and $f_s(r,s)$ can not vanish simultaneously.

4. The following parametric representation is due to Watson [23]:

$$C = a\,\mathrm{sn}\omega/\omega, \qquad c = a\,\mathrm{cn}\omega, \qquad b = a\,\mathrm{dn}\omega \tag{7.21}$$

where $\mathrm{sn}\omega$, $\mathrm{cn}\omega$, $\mathrm{dn}\omega$ are the Jacobian elliptic functions, $0 \leqq \omega \leqq K$, and the modulus

$$k = ((a^2 - b^2)/(a^2 - c^2))^{1/2}. \tag{7.22}$$

Thus the main inequalities (1.7) can be written as follows:

$$\tfrac{1}{3}\{(\mathrm{cn}\omega\mathrm{dn}\omega)^{1/2} + (\mathrm{cn}\omega)^{1/2} + (\mathrm{dn}\omega)^{1/2}\} \leqq \mathrm{sn}\omega/\omega \leqq \tfrac{1}{3}(1 + \mathrm{cn}\omega + \mathrm{dn}\omega), \tag{7.23}$$
$$0 \leqq \omega \leqq K.$$

5. The last inequality in (1.7) follows easily by use of the representation

$$M = \iint (a^2\xi^2 + b^2\eta^2 + c^2\zeta^2)^{1/2} d\omega \tag{7.24}$$

where the integration is extended over the unit sphere (ξ, η, ζ) with the surface element $d\omega$. Indeed

$$\begin{aligned} a\xi^2 + b\eta^2 + c\zeta^2 &\leqq (a^2\xi^2 + b^2\eta^2 + c^2\zeta^2)^{1/2}(\xi^2 + \eta^2 + \zeta^2)^{1/2} \\ &= (a^2\xi^2 + b^2\eta^2 + c^2\zeta^2)^{1/2}; \end{aligned} \tag{7.25}$$

integration over the unit sphere yields the desired inequality.

8. Ellipsoids of revolution. 1. *Prolate spheroid.* We have $b = c$, $\alpha = 0$, $\beta = \gamma$ so that according to a classical formula

$$A = 2\pi ab((1 - \beta^2)^{1/2} + \beta^{-1} \arcsin \beta); \tag{8.1}$$

on the other hand from (6.8)

$$C^{-1} = a^{-1}(2\beta)^{-1} \log \frac{1+\beta}{1-\beta} \tag{8.2}$$

and $b/a = (1 - \beta^2)^{1/2}$ so that

$$\begin{aligned} (R_A/C)^2 &= A/(4\pi C^2) \\ &= (2\beta)^{-3}(1 - \beta^2)^{1/2}(\beta(1 - \beta^2)^{1/2} + \arcsin \beta)\left(\log \frac{1+\beta}{1-\beta}\right)^2 = f(\beta). \end{aligned} \tag{8.3}$$

Now let

$$(8.4) \qquad f(z) = \sum_{n=0}^{\infty} c_{2n} z^{2n}, \qquad c_0 = 1,\ c_2 = c_4 = 0,\ c_6 = -4/945.$$

We prove that *the coefficients* $c_8, c_{10}, c_{12}, \cdots$ *are negative.*

The only singularities of the even function $f(z)$ in the finite part of the complex plane are ± 1. We have

$$(8.5) \qquad c_{2n} = (2\pi i)^{-1} \int f(z) z^{-2n-1} dz$$

where the path of the integration consists of the following parts:

(a) a "small" circle $|z-1| = \epsilon$ enclosing $+1$ in the negative sense;

(b) two straight lines joining $z = 1 + \epsilon$ and $z = \omega$ along the upper and lower border of the positive real axis;

(c) the "large" semicircle $|z| = \omega$ in the upper half plane;

(d) the curves corresponding to (a), (b), (c) obtained by the transformation $z' = -z$.

The contributions of the circles tend to 0 as $\epsilon \to 0$, $\omega \to \infty$ provided $n \geqq 1$. Now for $1 < z < +\infty$

$$(8.6) \qquad f(z) = (2z)^{-3} \cdot \mp i(z^2-1)^{1/2}$$
$$\times (\mp iz(z^2-1)^{1/2} + \pi/2 \pm i \log(z + (z^2-1)^{1/2})) \left(\log \frac{z+1}{z-1} \pm i\pi\right)^2$$

where the upper and lower sign corresponds to the upper and lower border. Here $(z^2-1)^{1/2}$ and the two logarithms are positive.

Hence for $n = 1, 2, 3, \cdots$

$$(8.7) \qquad c_{2n} = -16^{-1} \int_1^{\infty} (z^2-1)^{1/2} \left\{ \left(\log \frac{z+1}{z-1}\right)^2 - \pi^2 \right.$$
$$\left. + 4 \log \frac{z+1}{z-1} \left(z(z^2-1)^{1/2} - \log(z + (z^2-1)^{1/2})\right) \right\} z^{-2n-4} dz.$$

Let us write

$$(8.8) \qquad \begin{cases} \log \dfrac{z+1}{z-1} = u, \qquad 0 < u < +\infty, \\ \log \dfrac{z+1}{z-1} - \pi^2 \left(\log \dfrac{z+1}{z-1}\right)^{-1} + 4z(z^2-1)^{1/2} - 4\log(z + (z^2-1)^{1/2}) \\ = u - \pi^2 u^{-1} + 4 \coth(u/2)(\sinh(u/2))^{-1} - 4 \log \coth(u/4) = \phi(u). \end{cases}$$

Obviously $\lim_{u\to+0} \phi(u) = \lim_{u\to+\infty} \phi(u) = +\infty$. Furthermore

$$(8.9) \qquad (\sinh(u/2))^3\phi'(u) = (\sinh(u/2))^3 + \pi^2 u(u^{-1}\sinh(u/2))^3 - 4$$

is increasing from -4 to $+\infty$. Hence the function $\phi(u)$ is first decreasing and then increasing. Since $c_2 = c_4 = 0$ it can not remain always positive; thus it must change sign exactly twice. But according to a theorem of L. Fejér [cf. Pólya and Szegö **14**, vol. 2, pp. 50-51, problem 81] the sequence $c_2, c_4, c_6, c_8, c_{10}, \cdots$ can not contain more than two vanishing terms, or sign changes. Since $c_2 = c_4 = 0$ and $c_6 < 0$ all following terms must be negative.

At the same time we see that the coefficients of the power series expansion of C/R_A are all *positive*.

2. *Oblate spheroid.* In this case $a = b$, $\alpha = \beta$, $\gamma = 0$ and

$$(8.10) \qquad \begin{cases} A = 2\pi a^2(2\beta)^{-1}\left(2\beta + (1-\beta^2)\log\dfrac{1+\beta}{1-\beta}\right), \\ C^{-1} = a^{-1}\beta^{-1}\arcsin\beta \end{cases}$$

so that

$$(8.11) \qquad \begin{aligned} (R_A/C)^2 &= A/(4\pi C^2) \\ &= 2(2\beta)^{-3}(2\beta + (1-\beta^2)\log\frac{1+\beta}{1-\beta})(\arcsin\beta)^2 = f_1(\beta). \end{aligned}$$

Let

$$(8.12) \qquad f_1(z) = \sum_{n=0}^{\infty} c'_{2n} z^{2n}, \qquad c'_0 = 1,\ c'_2 = c'_4 = 0,\ c'_6 = 4/945.$$

We prove that *the coefficients $c'_8, c'_{10}, c'_{12}, \cdots$ are positive.*

With the same path of integration as in (8.5) we have

$$(8.13) \qquad c'_{2n} = (2\pi i)^{-1}\int f_1(z) z^{-2n-1}dz.$$

Now for $1 < z < +\infty$

$$(8.14) \qquad \begin{aligned} f_1(z) = 2(2z)^{-3}(2z + (1-z^2)\log\frac{z+1}{z-1} \pm i\pi(1-z^2)) \\ \times (\pi/2 \pm i\log(z + (z^2-1)^{1/2}))^2 \end{aligned}$$

on the upper and lower border, respectively. Hence

$$(8.15) \qquad \begin{aligned} c'_{2n} = 4^{-1}\int_1^{\infty}\Big\{\left(2z + (1-z^2)\log\frac{z+1}{z-1}\right)\log(z + (z^2-1)^{1/2}) \\ + (1-z^2)\left(\pi^2/4 - (\log(z+(z^2-1)^{1/2}))^2\right)\Big\} z^{-2n-4}dz. \end{aligned}$$

We write

$$(8.16)\quad \begin{cases} z = \cosh u,\ (z^2-1)^{1/2} = \sinh u,\ \log(z+(z^2-1)^{1/2}) = u,\ 0 < u < +\infty, \\ 2z(z^2-1)^{-1} - \log\dfrac{z+1}{z-1} - (\pi^2/4)\ (\log(z+(z^2-1)^{1/2}))^{-1} \\ \qquad\qquad + \log(z+(z^2-1)^{1/2}) \\ \quad = 2\cosh u(\sinh u)^{-2} - 2\log\coth(u/2) - \pi^2/(4u) + u = \phi_1(u). \end{cases}$$

Again $\lim_{u\to+0}\phi_1(u) = \lim_{u\to+\infty}\phi_1(u) = +\infty$ and

$$(8.17)\qquad (\sinh u)^3\phi'_1(u) = (\sinh u)^3 + (\pi^2/4)u(u^{-1}\sinh u)^3 - 4.$$

Thus we conclude as in the other case that $\phi_1(u)$ changes its sign exactly twice. The rest of the argument is the same as before.

9. Landen's transformation. By means of Landen's transformation we can define certain transformations of a given ellipsoid into another ellipsoid which preserve the capacity C.

1. Let

$$(9.1)\qquad a, b, c;\ \alpha, \beta, \gamma$$

have the same meaning as in Section **6** and let

$$(9.2)\qquad l, m, n;\ \lambda, \mu, \nu$$

be the corresponding quantities for another ellipsoid. Substituting in (6.4)

$$(9.3)\qquad u + a^2 = \beta^2a^2(\sin\phi)^{-2}$$

we obtain

$$(9.4)\qquad C^{-1} = (\beta a)^{-1}\int_0^\Phi \{1 - (\beta^{-1}\gamma)^2\sin^2\phi\}^{-1/2}d\phi$$

where

$$(9.5)\qquad \beta = \sin\Phi,\qquad 0 \leqq \Phi < \pi/2.$$

This is an elliptic integral of the first kind with modulus (7.22). According to Landen's transformation [3] it is equal to

$$(9.6)\quad 2(\beta a)^{-1}(1+\beta^{-1}\gamma)^{-1}\int_0^\Psi \{1 - 4\beta^{-1}\gamma(1+\beta^{-1}\gamma)^{-2}\sin^2\psi\}^{-1/2}d\psi$$

[3] See C. Jordan, *Cours d' Analyse*, 3rd Edition, Paris, 1913, vol. 2, pp. 117-118.

where the following relation holds:

$$\sin(2\Psi - \Phi) = \beta^{-1}\gamma \sin \Phi = \gamma. \tag{9.7}$$

We can identify the expression (9.6) with the reciprocal capacity of a second ellipsoid to which the quantities (9.2) correspond provided

$$2\mu l = a(\beta + \gamma), \qquad \mu = \sin \Psi, \qquad \mu^{-1}\nu = 2(\beta\gamma)^{1/2}(\beta + \gamma)^{-1}. \tag{9.8}$$

2. In order to find an explicit representation of l, m, n in terms of a, b, c we set

$$\beta = \sin u, \quad \gamma = \sin v, \qquad 0 \leqq v \leqq u < \pi/2, \tag{9.9}$$

so that from (9.5) and (9.7)

$$\Phi = u, \qquad \Psi = (u + v)/2. \tag{9.10}$$

Consequently in view of (9.8)

$$\mu = \sin((u + v)/2), \qquad \nu = (\sin u \sin v)^{1/2}\{\cos((u - v)/2)\}^{-1}. \tag{9.11}$$

Now from the first equation in (9.8)

$$l = a \cos((u - v)/2) \tag{9.12}$$

and

$$\begin{cases} m = l(1 - \nu^2)^{1/2} = a \cos((u + v)/2), \\ n = l(1 - \mu^2)^{1/2} = a \cos((u - v)/2) \cos((u + v)/2). \end{cases} \tag{9.13}$$

To sum up we have found:

$$\begin{cases} b = a \cos v, \quad c = a \cos u, \\ l = a \cos((u - v)/2), \ m = a \cos((u + v)/2), \\ n = a \cos((u - v)/2) \cos((u + v)/2). \end{cases} \tag{9.14}$$

Thus the following representation of our transformation arises:

$$\begin{cases} 2l = (a + b)^{1/2}(a + c)^{1/2} + (a - b)^{1/2}(a - c)^{1/2}, \\ 2m = (a + b)^{1/2}(a + c)^{1/2} - (a - b)^{1/2}(a - c)^{1/2}, \\ 2n = b + c. \end{cases} \tag{9.15}$$

3. From (9.14) and from (9.15) we conclude that

$$l < a, \quad m < b, \quad n > c \tag{9.16}$$

unless $b = c$ in which case $l = a$, $m = b$, $n = c$. Repeated application of the

transformation (9.15) leads to a sequence of ellipsoids with the same capacity C to which the quantities

$$(9.17)\qquad a_h, b_h, c_h;\ \alpha_h, \beta_h, \gamma_h$$

correspond; (9.17) is identical with (9.1) for $h = 0$ and with (9.2) for $h = 1$. Since the sequences $\{a_h\}$ and $\{b_h\}$ are decreasing and the sequence $\{c_h\}$ is increasing the limits

$$(9.18)\qquad \lim_{h\to\infty} a_h = a^*, \qquad \lim_{h\to\infty} b_h = b^*, \qquad \lim_{h\to\infty} c_h = c^*$$

exist.

Obviously $a \geqq a^* \geqq b^* \geqq c^* \geqq c$. We prove that $b^* = c^*$, that is *the limiting case is always a prolate ellipsoid of revolution.* Indeed let $0 < \gamma < \beta < 1$; from the third equation in (9.8) we have $1 > \mu^{-1}\nu > (\beta^{-1}\gamma)^{\frac{1}{2}}$ so that

$$(9.19)\qquad 1 > \beta_h^{-1}\gamma_h > (\beta^{-1}\gamma)^{\frac{1}{2}h}.$$

This yields $\lim_{h\to\infty} \beta_h^{-1}\gamma_h = 1$, and shows the rapidity of the convergence.

From (9.15) it is obvious that if the initial ellipsoid is an oblate spheroid the same will be true of all the ellipsoids of the sequence derived from it. In this case the limiting ellipsoid is a sphere and our process reduces to a well known algorithm for the computation of the arc sin function [cf. (8.10)]. Prolate spheroids remain unchanged under the transformation (9.15).

Finally, based on Watson's formulas (7.21) and by the use of the Jacobian elliptic functions, the following representation of the limits (9.18) can be obtained:

$$(9.20)\qquad a^* = \pi C/(2K')\coth(\pi/(2K')),\ b^* = c^* = \pi C/(2K')\{\sinh(\pi/(2K'))\}^{-1};$$

here K' is the complete elliptic integral of the first kind associated with the complementary modulus

$$(9.21)\qquad k' = ((b^2 - c^2)/(a^2 - c^2))^{\frac{1}{2}}.$$

4. An interesting property of the transformation (9.15) is the following:

The transformation (9.15) *diminishes the sum and increases the product of the semi-axes, the only exception being the case of a prolate spheroid.*

Using (9.14) we have indeed

$$\begin{cases} 1 + \cos u + \cos v \geqq \cos((u+v)/2) + \cos((u-v)/2) \\ \qquad\qquad + \cos((u+v)/2)\cos((u-v)/2), \\ \cos u \cos v \leqq \{\cos((u+v)/2)\cos((u-v)/2)\}^2 \end{cases}$$

with equality only for $u = v$.

In view of this remark the proof of the inequalities

$$(abc)^{1/3} \leqq C \leqq \tfrac{1}{3}(a + b + c)$$

could be reduced to the special case of a prolate ellipsoid of revolution. A less simple argument shows that the quantity $(bc)^{1/2} + (ca)^{1/2} + (ab)^{1/2}$ is increased by the transformation (9.15).

10. Further special cases.[4] 1. Let s_0 and s_1 be two closed curves in the x, y-plane, s_1 interior to s_0. We consider an arbitrary continuous function ψ with piece-wise continuous derivatives defined between s_0 and s_1, $\psi = 0$ on s_0 and $\psi = 1$ on s_1. The minimum γ of the integral

$$(4\pi)^{-1} \int\int |\operatorname{grad} \psi|^2 dxdy \tag{10.1}$$

the integration being extended over the set of points between s_0 and s_1, is the "capacity per unit length" of a condenser formed by two infinite cylinders S_0 and S_1 whose orthogonal cross-section are s_0 and s_1. (Cf. Section **4**.)

We consider a condenser formed by the infinite cylinder S_0 as "exterior surface" and by a *finite* part of S_1, obtained by intersecting the infinite cylinder S_1 orthogonally by two planes at a given distance l from each other. To the finite cylinder we add the two bases so that the "interior surface" is the complete boundary of a finite solid cylinder of height l. The two cylinders have parallel generators. We denote the capacity of this condenser by C_{if}.

A modified form of the previous condenser arises by intersecting also the cylinder S_0 by the planes mentioned, however without considering the bases of the exterior cylinder. In this case the exterior surface is a finite part of a cylinder (without bases) and the interior surface is, as before, the complete boundary of a solid cylinder. We denote the capacity of this condenser by C_{ff}.

Finally we may remove the bases of the interior cylinder and consider the condenser formed by the two finite cylindrical surfaces constructed before, both without the bases. Let C denote the capacity of this condenser.

It is interesting to compare the quantities

$$l\gamma,\ C_{if},\ C_{ff},\ C. \tag{10.2}$$

We prove the following inequalities:

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[4] Added November 16, 1944.

$$(10.3) \qquad l\gamma < C < C_{ff} < C_{if} < l\gamma + (\gamma A/\pi)^{1/2}$$

where A is the area of the curve s_0.

First we observe that C_{if} is the minimum of the integral

$$(10.4) \qquad (4\pi)^{-1} \int\int\int |\operatorname{grad} \Psi|^2 \, dxdydz$$

where $\Psi(x, y, z)$ is an arbitrary continuous function with piece-wise continuous derivatives in the field between S_0 and the finite cylinder, $\Psi = 0$ on S_0 and $\Psi = 1$ on the finite cylinder. Extending the definition of Ψ to the space exterior to S_0 where we set $\Psi = 0$, we obtain an admissible function for the corresponding minimum problem defining C_{ff}. Finally, extending the definition of any function Ψ^* admissible in this latter problem to the space in the interior of the finite inner cylinder where we set $\Psi^* = 1$, we obtain an admissible function for the minimum problem defining C. This proves the inequalities connecting C_{if}, C_{ff} and C.

As to the lower bound for C we denote by $\Psi_0(x, y, z)$ the function for which the minimum C is attained:

$$(10.5) \qquad C = (4\pi)^{-1} \int\int\int |\operatorname{grad} \Psi_0|^2 \, dxdydz.$$

This integral will be diminished by reducing the infinite domain of integration to the finite domain bounded by the given finite cylinders and the two orthogonal planes mentioned. The integral will be further diminished by omitting from $|\operatorname{grad} \Psi_0|^2$ the term $(\partial\Psi_0/\partial z)^2$ so that

$$(10.6) \qquad C > (1/4\pi) \int_0^l \{ \int\int [(\partial\Psi_0/\partial x)^2 + (\partial\Psi_0/\partial y)^2] dxdy \} dz.$$

(We assume that the perpendicular cross-sections are $z = 0$ and $z = l$.) For each fixed value of z, $0 < z < l$, the function $\Psi_0(x, y, z)$ is an admissible function for the minimum problem (10.1) so that the interior integral is not less than $4\pi\gamma$. From this the assertion follows.

In order to prove the upper bound for C_{if}, let $\psi_0(x, y)$ denote the function for which the minimum of (10.1) is attained. We define $\psi_0(x, y) = 1$ in the interior of s_1. If $h(z)$ is an arbitrary continuous function of z satisfying the conditions

$$\begin{cases} h(z) = 1 \quad \text{for} \quad 0 \leqq z \leqq l, \\ \lim_{z \to \pm\infty} h(z) = 0, \end{cases}$$

the function $\psi_0(x,y)h(z)$ will be admissible for the minimum problem defining C_{if}. Consequently

$$C_{if} < (1/4\pi)\int_{-\infty}^{\infty}\{\iint[(\partial\psi_0/\partial x)^2+(\partial\psi_0/\partial y)^2]dxdy\}(h(z))^2dz$$
$$+ (1/4\pi)\int_{-\infty}^{\infty}\{\iint\psi_0^2dxdy\}(h'(z))^2dz.$$

In the first integral with respect to x and y we integrate only in the ring-shaped domain between the curves s_0 and s_1; this yields $4\pi\gamma$. The second integral with respect to x and y will be less than A. Hence

$$(10.7)\quad \begin{aligned} C_{if} &< \gamma\int_{-\infty}^{\infty}(h(z))^2dz+(A/4\pi)\int_{-\infty}^{\infty}(h'(z))^2dz \\ &= l\gamma+\gamma\int_{-\infty}^{0}(h(z))^2dz+(A/4\pi)\int_{-\infty}^{0}(h'(z))^2dz \\ &\quad +\gamma\int_{l}^{\infty}(h(z))^2dz+(A/4\pi)\int_{l}^{\infty}(h'(z))^2dz. \end{aligned}$$

Writing, with $\lambda > 0$,

$$(10.8)\quad \begin{cases} h(z)=e^{\lambda z}, & z\leqq 0,\\ h(z)=e^{-\lambda(z-l)}, & z\geqq l,\end{cases}$$

we obtain

$$(10.9)\quad C_{if} < l\gamma+\gamma/\lambda+A\lambda/(4\pi).$$

The most advantageous value λ is $(4\pi\gamma/A)^{1/2}$; this furnishes the asserted inequality.

2. Of all closed curves which enclose a given area, the circle has the smallest outer radius (transfinite diameter). This is the area theorem of Bieberbach (see Section 1).

Of all triangles with given area, the equilateral triangle has the smallest outer radius.

First, the outer radius is a continuous function of the coordinates of the vertices of the triangle. Indeed, let Δ be a given triangle, k a positive constant. We denote by $k\Delta$ the triangle which arises by changing the sides of Δ in the ratio $1:k$. If $\bar{r}$ is the outer radius of Δ, then the outer radius of $k\Delta$ is $k\bar{r}$. Now let ϵ be an arbitrary positive number, $\epsilon < 1$. If the vertices of a triangle Δ_1 are sufficiently near to those of Δ, the triangle Δ_1 can be enclosed between the triangles $(1-\epsilon)\Delta$ and $(1+\epsilon)\Delta$. Hence the assertion follows.

(The same argument shows that the outer radius of an arbitrary closed

polygon is a continuous function of the coordinates of the vertices, provided the polygon is convex or, more generally, star-shaped with respect to a certain point.)

Consequently, there must exist a triangle with given area and with a smallest outer radius. If this triangle were not equilateral at least two sides would be different. Symmetrizing with respect to a line perpendicular to the third side, the outer radius would diminish. This is a contradiction so that the triangle with the minimum property must be equilateral.

Of all quadrilaterals with given area, the square has the smallest outer radius.

Let $ABCD$ be a quadrilateral with the smallest outer radius. If two adjacent sides were different, say $AB \neq BC$, we could symmetrize with respect to a line perpendicular to the diagonal AC, and the outer radius would diminish. Hence the minimum can be attained only for quadrilaterals with equal sides. If such a quadrilateral were not a square, symmetrizing with respect to a line perpendicular to a side, we could diminish the outer radius. Consequently, the quadrilateral with the minimum property must be a square.

Of all tetrahedra with given volume, the regular tetrahedron has the smallest capacity.

We symmetrize in this case with respect to a plane perpendicular to one of the edges.

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the function $\psi_0(x, y)h(z)$ will be admissible for the minimum problem defining C_{if}. Consequently

$$C_{if} < (1/4\pi)\int_{-\infty}^{\infty}\{\int\int[(\partial\psi_0/\partial x)^2 + (\partial\psi_0/\partial y)^2]dxdy\}(h(z))^2dz$$
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ON FABER POLYNOMIALS.*

By Issai Schur.[1]

I. Introduction.[2] Let

$$(1)\quad f(z) = z + a_1 + a_2/z + a_3/z^2 + \cdots = z\sum_{\nu=0}^{\infty} a_\nu z^{-\nu} = zg(1/z), \quad a_0 = 1$$

be a power series concerning the convergence of which no assumption is made.[3]

* Received November 8, 1943; Revised February 4, 1944.

[1] Died January 10, 1941, in Tel Aviv, Palestine. The Einstein Institute of Mathematics of the Hebrew University, Jerusalem, has undertaken the complete edition of the posthumous papers of the deceased, its honorary member since 1940. As the realization of this project under present conditions requires considerable time, some of the main results of this scientific legacy will be published in preliminary notes. The present note has been elaborated by Dr. M. Schiffer of the Hebrew University who worked over the notes left on the subject in cooperation with Professor M. Fekete, the general editor of the scientific legacy of the great scholar. The manuscript has been revised in this country.

[2] Grunsky gave necessary and sufficient conditions for the coefficients of a function in order that it be meromorphic and univalent in a given domain D. ("Koeffizientenabschätzungen für schlicht abbildende meromorphe Funktionen," *Mathematische Zeitschrift*, vol. 45 (1939), pp. 29-61). If, in particular, D is the exterior of the unit circle, these conditions take the form

$$\text{(i)}\qquad \Big|\sum_{\mu,\nu=1}^{m} \nu c_{\mu\nu} x_\mu x_\nu\Big| \leqq \sum_{\nu=1}^{m} \nu\,|x_\nu|^2, \qquad (m = 1, 2, \cdots),$$

where the $c_{\mu\nu}$ are defined by the formula (2) of this paper, if the function considered has the form (1). The identity $\nu c_{\mu\nu} = \mu c_{\nu\mu}$ is proved by Grunsky with the aid of Cauchy's residue theorem. The late Professor Schur wanted to bring the conditions (i) into a more easily evaluable form and investigated, therefore, the relations between the coefficients a_ν and the $c_{\mu\nu}$. This paper gives the results he obtained. Another paper, caused by the same problem, dealing with the transformation of quadratic forms to principal axes will appear elsewhere.

[3] In the formal algebra of power series, two series are called equal if corresponding coefficients are identical. We define the sum of $P(x) = \sum_{\nu=a}^{\infty} k_\nu x^\nu$ $(a > -\infty)$ and $P^*(x) = \sum_{\nu=a}^{\infty} k^*_\nu x^\nu$ to be the series $P(x) + P^*(x) = \sum_{\nu=a}^{\infty} (k_\nu + k^*_\nu) x^\nu$ and the product $P(x)P^*(x)$ to be $\sum_{\nu=2a}^{\infty} l_\nu x^\nu$ with $l_\nu = \sum_{\rho=a}^{\nu-a} k_\rho k^*_{\nu-\rho}$. Finally $P(x)^{-1}$ is the power series which satisfies $P(x)P(x)^{-1} = 1$, and the derivative $P'(x)$ of $P(x)$ is $\sum_{\nu=a}^{\infty} \nu k_\nu x^{\nu-1}$.

We define a polynomial $P_m(f)$ in $f(z)$ of degree m $(m = 1, 2, \cdots)$ such that

$$(2) \quad P_m(f) = z^m + c_{m1}/z + c_{m2}/z^2 + \cdots + c_{m\mu}/z^\mu + \cdots = z^m + G_m(1/z),$$

$$G_m(x) = \sum_{\mu=1}^{\infty} c_{m\mu} x^\mu.$$

$P_m(f)$ is called the m-th Faber polynomial of $f(z)$. The existence and uniqueness of $P_m(f)$ for $m \geqq 1$ is easily shown by recursion.

Let

$$(3) \qquad Q(f) = q_0 z^m + q_1 z^{m-1} + \cdots + q_m + q'/z + \cdots$$

be any polynomial in $f(z)$ of degree m. Then, writing $P_0(f) = 1$,

$$D(f) = Q(f) - q_0 P_m(f) - q_1 P_{m-1}(f) - \cdots - q_m P_0(f) = \alpha/z + \cdots$$

is a polynomial in $f(z)$ the development of which with respect to z contains only negative powers. This being evidently impossible unless $D(f)$ is identically zero, we have the development

$$(3') \qquad Q(f) = q_0 P_m(f) + q_1 P_{m-1}(f) + \cdots + q_m P_0(f).$$

Letting

$$(4) \qquad g(x)^m = \sum_{\mu=0}^{\infty} a_{m\mu} x^\mu \qquad (m = 1, 2, \cdots), \quad a_{m0} = 1$$

and writing $x = 1/z$ we have

$$f(z)^m = z^m g(x)^m = z^m + a_{m1} z^{m-1} + a_{m2} z^{m-2} + \cdots + a_{mm} + a_{m,m+1}/z + \cdots$$

whence, according to (3) and (3′),

$$(5) \quad f(z)^m = P_m(f) + a_{m1} P_{m-1}(f) + \cdots + a_{m,m-1} P_1(f) + a_{mm} P_0(f).$$

Let $\phi_m(x) = 1 + a_{m1} x + \cdots + a_{mm} x^m$ and $\psi_m(x) = a_{m,m+1} x + \cdots + a_{m,m+\nu} x^\nu + \cdots$. Then $f(z)^m = z^m \phi_m(x) + \psi_m(x)$ and therefore, by (2) and (5),

$$(6) \qquad \psi_m(x) = G_m(x) + a_{m1} G_{m-1}(x) + \cdots + a_{m,m-1} G_1(x).$$

This important identity establishes a relation between the coefficients $c_{\mu\nu}$ defined in (2) and the $a_{\mu\nu}$ defined in (4). In fact, comparing coefficients of like powers of x, we have for $\nu \geqq 1$, $m \geqq 1$,

$$(7) \qquad a_{m,m+\nu} = c_{m\nu} + a_{m1} c_{m-1,\nu} + a_{m2} c_{m-2,\nu} + \cdots + a_{m,m-1} c_{1\nu}.$$

In order to combine all these formulas in one, we introduce the infinite matrices

$$(8)\quad \begin{cases} A = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ a_{21} & 1 & 0 & \cdots \\ a_{32} & a_{31} & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{pmatrix} = (a_{\mu,\mu-\nu}), \quad a_{\mu 0} = 1, \quad a_{\mu,-k} = 0 \text{ for } k \geqq 1, \\ B = \begin{pmatrix} a_{12} & a_{13} & \cdots \\ a_{23} & a_{24} & \cdots \\ a_{34} & a_{35} & \cdots \\ \cdot & \cdot & \cdots \end{pmatrix} = (a_{\mu,\mu+\nu}), \qquad C = \begin{pmatrix} c_{11} & c_{12} & \cdots \\ c_{21} & c_{22} & \cdots \\ \cdot & \cdot & \cdots \end{pmatrix} = (c_{\mu\nu}). \end{cases}$$

Then (7) can be expressed in the equivalent forms

$$(7') \qquad B = AC, \qquad C = A^{-1}B.$$

With the aid of (7′) we shall give an *explicit formula for the $c_{\mu\nu}$ in terms of the coefficients a_ν of $f(z)$*. We shall see that each $c_{\mu\nu}$ is a polynomial in the a_ν with non-negative integer coefficients, and that $\nu c_{\mu\nu} = \mu c_{\nu\mu}$ (Grunsky's identity). This can also be shown by other arguments [4] but we shall calculate the coefficients of these polynomials explicitly, and shall see in particular that Grunsky's formula is an expression of a corresponding symmetry property of the polynomial coefficients.

II. Computation of the elements of the matrix *C*. We define, in conformity with (4),

$$(4') \qquad g(x)^{-m} = \sum_{\mu=0}^{\infty} a_{-m,\mu} x^{\mu}, \qquad (m = 1, 2, \cdots), \; a_{-m,0} = 1.$$

In particular, we have in $a_{-1,\mu} = p_\mu$ the well-known Aleph-functions of Wronski. In order to establish relations between the $a_{-m,\mu}$ and the $a_{n\mu}$, we make use of the following simple lemma:

LEMMA. *Let $g(x) = \sum_{\nu=0}^{\infty} a_\nu x^\nu$ be an arbitrary power series. Then*

$$[g(x)^k - xg'(x)g(x)^{k-1}]_k = 0$$

where $[u(x)]_k$ denotes the coefficient of x^k in the development of $u(x)$ in powers of x.

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[4] The integral character of the coefficients follows immediately by induction from (7) since i) $a_{m,\mu}$ (by (1) and (4)) is a polynomial in a_ν with integral coefficients for $m \geqq 1$, $\mu \geqq 0$; ii) $c_{1\nu} = a_{\nu+1}$ for $\nu \geqq 1$ by (1) and (2).

The truth of the lemma is evident since

$$g(x)^k = \sum_{\rho=0}^{\infty} a_{k\rho} x^\rho \quad \text{and} \quad x g'(x) g(x)^{k-1} = \sum_{\rho=0}^{\infty} (\rho/k) a_{k\rho} x^\rho.$$

We apply the lemma with $k = \mu - \nu$, μ and ν $(\nu < \mu)$ being arbitrary positive integers, and obtain

$$0 = [g(x)^{\mu-\nu} - x g'(x) g(x)^{\mu-\nu-1}]_{\mu-\nu} = \left[g(x)^\mu \left(\frac{1}{g(x)^\nu} - \frac{x g'(x)}{g(x)^{\nu+1}}\right)\right]_{\mu-\nu}$$

$$= \left[\frac{g(x)^\mu}{\nu x^{\nu-1}} \left(\frac{x^\nu}{g(x)^\nu}\right)'\right]_{\mu-\nu} = \left[\frac{g(x)^\mu}{\nu} \sum_{\lambda=0}^{\infty} (\lambda+\nu) a_{-\nu,\lambda} x^\lambda\right]_{\mu-\nu}.$$

Hence

$$(9) \qquad a_{\mu,\mu-\nu} + \frac{\nu+1}{\nu} a_{\mu,\mu-\nu-1} a_{-\nu,1} + \frac{\nu+2}{\nu} a_{\mu,\mu-\nu-2} a_{-\nu,2} + \cdots + \frac{\mu}{\nu} a_{-\nu,\mu-\nu} = 0,$$

which, by (8) and (4′), yields

$$(10) \qquad A^{-1} = \left(\frac{\mu}{\nu} a_{-\nu,\mu-\nu}\right), \qquad a_{-\nu,-k} = 0 \text{ for } k \geqq 1.$$

From (7′) and (10) we obtain the formula

$$(11) \qquad c_{\mu\nu} = \sum_{\lambda=1}^{\mu} \frac{\mu}{\lambda} a_{-\lambda,\mu-\lambda} a_{\lambda,\lambda+\nu}$$

as a starting point for further calculations.

We begin by computing $a_{-1,\mu} = p_\mu$, for which we obtain the well-known formula

$$(12) \qquad p_\mu = \sum (-1)^{\alpha_1+\alpha_2+\cdots+\alpha_\mu} \frac{(\alpha_1+\alpha_2+\cdots+\alpha_\mu)!}{\alpha_1!\,\alpha_2!\cdots\alpha_\mu!} a_1^{\alpha_1}\cdots a_\mu^{\alpha_\mu}$$
$$(\alpha_1 + 2\alpha_2 + \cdots + \mu\alpha_\mu = \mu).$$

Differentiating the identity $g(x)^{-1} = \sum_{\mu=0}^{\infty} p_\mu x^\mu$ $\lambda - 1$ times with respect to a_1 we have

$$(13) \qquad (-1)^{\lambda-1} \frac{(\lambda-1)!\, x^{\lambda-1}}{g(x)^\lambda} = \sum_{\mu=0}^{\infty} \frac{\partial^{\lambda-1} p_\mu}{\partial a_1^{\lambda-1}} x^\mu.$$

Hence by (4′)

$$(14) \qquad a_{-\lambda,\mu-\lambda} = (-1)^{\lambda-1} \frac{1}{(\lambda-1)!} \frac{\partial^{\lambda-1} p_{\mu-1}}{\partial a_1^{\lambda-1}}$$

and so by (12)

$$(15) \qquad a_{-\lambda,\mu-\lambda} = \sum_{\alpha_1+2\alpha_2+\cdots+(\mu-1)\alpha_{\mu-1}=\mu-1} (-1)^{\lambda-1+\alpha_1+\cdots+\alpha_{\mu-1}}$$
$$\times \frac{(\alpha_1+\alpha_2+\cdots\alpha_{\mu-1})!}{\alpha_1!\,\alpha_2!\cdots\alpha_{\mu-1}!} \binom{\alpha_1}{\lambda-1} a_1^{\alpha_1-(\lambda-1)} a_2^{\alpha_2}\cdots a_{\mu-1}^{\alpha_{\mu-1}}.$$

The m-th Faber polynomial $P^*_m(f^*)$ of $f^*(z)=f(z)+c$ is evidently connected with the m-th Faber polynomial $P_m(f)$ of $f(z)$ by the relation $P^*_m(f^*)=P_m(f^*-c)$ which, since $P_m(f^*-c)=P_m(f)$, shows that the matrices C associated according to (2) and (8) with $f(z)$ and $f^*(z)$ are the same and thus do not depend on a_1. For our final aim, to compute the elements $c_{\mu\nu}$ of C, we may, therefore, assume henceforth that $a_1=0$. The coefficients which correspond to this assumption will be denoted $a_{ik}{}^{(0)}$.

From (15) (with $a_1=0$) we have

$$(16)\quad a^{(0)}_{-\lambda,\mu-\lambda}=\sum(-1)^{\alpha_2+\alpha_3+\cdots+\alpha_{\mu-\lambda}}\frac{(\lambda-1+\alpha_2+\cdots\alpha_{\mu-\lambda})!}{(\lambda-1)!\,\alpha_2!\cdots\alpha_{\mu-\lambda}!}a_2{}^{\alpha_2}\cdots a_{\mu-\lambda}^{\alpha_{\mu-\lambda}}$$

$$(2\alpha_2+\cdots+(\mu-\lambda)\alpha_{\mu-\lambda}=\mu-\lambda).$$

Also

$$(17)\quad a^{(0)}_{\lambda,\mu+\lambda}=\sum\frac{\lambda!}{(\lambda-\beta_2-\beta_3-\cdots)!\,\beta_2!\beta_3!\cdots\beta_{\mu+\lambda}!}a_2{}^{\beta_2}a_3{}^{\beta_3}\cdots a_{\mu+\lambda}^{\beta_{\mu+\lambda}}$$

$$(2\beta_2+\cdots+(\mu+\lambda)\beta_{\mu+\lambda}=\mu+\lambda).$$

Introducing (16) and (17) into (11) we get

$$(18)\quad c_{\mu\nu}=\sum_{\lambda=1}^{\mu}(\mu/\lambda)\sum_{A=\mu-\lambda}(-1)^{\alpha}\frac{(\lambda-1+\alpha)!\,a_2{}^{\alpha_2}a_3{}^{\alpha_3}\cdots a_{\mu-\lambda}^{\alpha_{\mu-\lambda}}}{(\lambda-1)!\,\alpha_2!\,\alpha_3!\cdots\alpha_{\mu-\lambda}!}$$

$$\times\sum_{B=\lambda+\nu}\frac{\lambda!\,a_2{}^{\beta_2}a_3{}^{\beta_3}\cdots a_{\lambda+\nu}^{\beta_{\lambda+\nu}}}{(\lambda-\beta)!\,\beta_2!\,\beta_3!\cdots\beta_{\lambda+\nu}!}$$

where the abbreviations $\alpha=\alpha_2+\alpha_3+\cdots+\alpha_{\mu-\lambda}$, $\beta=\beta_2+\beta_3+\cdots+\beta_{\lambda+\nu}$, $A=2\alpha_2+3\alpha_3+\cdots+(\mu-\lambda)\alpha_{\mu-\lambda}$, $B=2\beta_2+3\beta_3+\cdots+(\lambda+\nu)\beta_{\lambda+\nu}$ have been introduced. From (18) we see that $c_{\mu\nu}$ has degree $[\frac{1}{2}(\mu+\nu)]$ (at most) and weight $\mu+\nu$.

Let

$$(19)\quad c_{\mu\nu}=\sum_{\Gamma=\mu+\nu}C^{(\mu,\nu)}_{\gamma_2\gamma_3\cdots\gamma_{\mu+\nu}}a_2{}^{\gamma_2}a_3{}^{\gamma_3}\cdots a_{\mu+\nu}^{\gamma_{\mu+\nu}},$$

$$\gamma=\gamma_2+\cdots+\gamma_{\mu+\nu},\quad \Gamma=2\gamma_2+\cdots+(\mu+\nu)\gamma_{\mu+\nu}.$$

We have now to compute the integers $C^{(\mu,\nu)}_{\gamma_2\gamma_3\cdots\gamma_{\mu+\nu}}$. From (18) and (19) we obtain

$$(20)\quad C^{(\mu,\nu)}_{\gamma_2\gamma_3\cdots\gamma_{\mu+\nu}}=\sum_{\lambda=1}^{\mu}\mu\sum_{A=\mu-\lambda}(-1)^{\alpha}\frac{(\lambda-1+\alpha)!}{\gamma_2!\cdots\gamma_{\mu+\nu}!\,(\lambda-\gamma+\alpha)!}$$

$$\times\binom{\gamma_2}{\alpha_2}\cdots\binom{\gamma_{\mu-\lambda}}{\alpha_{\mu-\lambda}}=\frac{\mu(\gamma-1)!}{\gamma_2!\cdots\gamma_{\mu+\nu}!}\sum_{\lambda=1}^{\mu}\sum_{A=\mu-\lambda}(-1)^{\alpha}\binom{\lambda-1+\alpha}{\gamma-1}\binom{\gamma_2}{\alpha_2}\cdots\binom{\gamma_{\mu-\lambda}}{\alpha_{\mu-\lambda}}.$$

Taking into consideration that in (20) the summation indices $\lambda, \alpha_2, \cdots, \alpha_{\mu+\nu}$ are always connected by the equation $\lambda = \mu - A$, we may transform it into the form

$$(21) \quad C^{(\mu,\nu)}_{\gamma_2\gamma_3\ldots\gamma_{\mu+\nu}} = \frac{\mu(\gamma-1)!}{\gamma_2!\,\gamma_3!\cdots\gamma_{\mu+\nu}!} \sum (-1)^\alpha \binom{\gamma_2}{\alpha_2}\binom{\gamma_3}{\alpha_3}\cdots\binom{\gamma_{\mu-1}}{\alpha_{\mu-1}}\binom{\mu-1-A+\alpha}{\gamma-1}$$

where the summation is to be extended over all non-negative integer values of α_i, the symbol $\binom{u}{v}$ being defined in the usual way for $u \geqq v$ and as 0 for $u < v$ even if u is negative. Thus we have to calculate only the expressions

$$(22) \quad D^{(\mu)}_{\gamma_2\gamma_3\ldots\gamma_{\mu+\nu}} = \sum (-1)^\alpha \binom{\gamma_2}{\alpha_2}\binom{\gamma_3}{\alpha_3}\cdots\binom{\gamma_{\mu-1}}{\alpha_{\mu-1}}\binom{\mu-1-A+\alpha}{\gamma-1}.$$

Since (with our convention concerning $\binom{u}{v}$) the expression $\binom{\mu-1-A+\alpha}{\gamma-1}$ vanishes unless $\mu - A + \alpha \geqq \gamma$, that is unless $\mu - \gamma \geqq \alpha_2 + 2\alpha_3 + \cdots + (\mu+\nu-1)\alpha_{\mu+\nu}$, we see that $\alpha_\mu = \alpha_{\mu+\nu} = \cdots = \alpha_{\mu+\nu} = 0$, and so we have

$$(23) \quad D^{(\mu)}_{\gamma_2\ldots\gamma_{\mu+\nu}} = \sum (-1)^\alpha \binom{\gamma_2}{\alpha_2}\cdots\binom{\gamma_{\mu+\nu}}{\alpha_{\mu+\nu}}\binom{\mu-1-A+\alpha}{\gamma-1}$$

where α_i again takes only non-negative integer values and

$$\alpha = \alpha_2 + \alpha_3 + \cdots + \alpha_{\mu+\nu}, \qquad A = 2\alpha_2 + 3\alpha_3 + \cdots + (\mu+\nu)\alpha_{\mu+\nu}.$$

III. The explicit formula for $c_{\mu\nu}$. The expression (23) can be summed successively with the aid of the following lemma.

LEMMA. *Let m and n be integers, $m \geqq 1$, $n \geqq 0$. Let $b^{(m)}_{n,k} = 0$ for $k > n(m-1)$ or $k < 0$, and let $b^{(m)}_{n,k}$ be defined for $0 \leqq k \leqq n(m-1)$ by*

$$(24) \quad \left(\frac{1-x^m}{1-x}\right)^n = \sum_{k=0}^{n(m-1)} b^{(m)}_{n,k}\, x^k.$$

(Thus $b^{(m)}_{n,k}$ is a non-negative integer for $m \geqq 1$, $n \geqq 0$, and k arbitrary). *Then* (assuming the above convention concerning $\binom{u}{v}$) *we have for arbitrary positive integers h and r the identities*

$$(25)\quad \sum_{\nu=0}^{n}(-1)^{\nu}\binom{n}{\nu}\binom{h+n-1+r-m\nu}{h+n-1}\equiv\sum_{\rho=0}^{r}b_{n,\rho}^{(m)}\binom{h-1+r-\rho}{h-1}$$
$$=\sum_{\rho=-\infty}^{+\infty}b_{n,\rho}^{(m)}\binom{h-1+r-\rho}{h-1}.$$

((25) is trivially true for $h>0$, $r\leqq 0$).

We have (by the binomial theorem)

$$(26)\qquad (1-x)^{-(h+n)}=\sum_{\mu=0}^{\infty}\binom{h+n-1+\mu}{h+n-1}x^{\mu}$$

whence

$$(27)\qquad \left[\frac{(1-x^m)^n}{(1-x)^{h+n}}\right]_r=\sum_{\mu+m\nu=r}^{\mu,\nu\geqq 0}(-1)^{\nu}\binom{n}{\nu}\binom{h+n-1+\mu}{h+n-1}$$
$$=\sum_{\nu=0}^{n}(-1)^{\nu}\binom{n}{\nu}\binom{h+n-1+r-m\nu}{h+n-1}.$$

On the other hand, by (24),

$$(28)\qquad \left[\frac{(1-x^m)^n}{(1-x)^{h+n}}\right]_r=\left[\sum_{\rho=0}^{n(m-1)}b_{n,\rho}^{(m)}x^{\rho}\sum_{\sigma=0}^{\infty}\binom{h-1+\sigma}{h-1}x^{\sigma}\right]_r$$
$$=\sum_{\rho+\sigma=r}^{\rho,\sigma\geqq 0}b_{n,\rho}^{(m)}\binom{h-1+\sigma}{h-1}=\sum_{\rho=0}^{r}b_{n,\rho}^{(m)}\binom{h-1+r-\rho}{h-1}.$$

Comparing (27) and (28) we obtain (25).

In the case $h=0$, $n>0$, combination of (24) with (27) yields the additional equality

$$(29)\qquad \sum_{\nu=0}^{n}(-1)^{\nu}\binom{n}{\nu}\binom{n-1+r-m\nu}{n-1}=b_{n,r}^{(m)}.$$

To carry out the summation in (23) we apply (25) and (29). Let γ_j $(2\leqq j\leqq \mu+\nu)$ be the last non-vanishing term in $\gamma_2,\cdots,\gamma_{\mu+\nu}$. If $j=2$, then by (23) and (29) (and the convention about $\binom{u}{v}$)

$$(30)\quad D_{\gamma_2\ldots\gamma_{\mu+\nu}}^{(\mu)}=\sum_{\alpha_2}(-1)^{\alpha_2}\binom{\gamma_2}{\alpha_2}\binom{\mu-1-\alpha_2}{\gamma_2-1}=b_{\gamma_2,\mu-\gamma_2}^{(1)}.$$

If $j\geqq 3$ we set $\bar{\alpha}=\alpha_3+\cdots+\alpha_j$, $\bar{A}=3\alpha_3+\cdots+j\alpha_j$, $\bar{\gamma}=\gamma_3+\cdots+\gamma_j$, and obtain

$$(31)\quad D_{\gamma_2\ldots\gamma_{\mu+\nu}}^{(\mu)}=\sum_{\alpha_3,\ldots,\alpha_j}(-1)^{\bar{\alpha}}\binom{\gamma_3}{\alpha_3}\cdots\binom{\gamma_j}{\alpha_j}$$
$$\sum_{\alpha_2}(-1)^{\alpha_2}\binom{\gamma_2}{\alpha_2}\binom{\mu-1-\bar{A}+\bar{\alpha}-\alpha_2}{\bar{\gamma}+\gamma_2-1}.$$

The inner sum can be evaluated by applying (25) with $\nu = \alpha_2$, $n = \gamma_2$, $h = \bar{\gamma}$, $m = 1$, $r = \mu - \gamma - \bar{A} + \bar{\alpha}$. We obtain

$$(32) \quad \sum_{\alpha_2=0}^{\gamma_2} (-1)^{\alpha_2} \binom{\gamma_2}{\alpha_2} \binom{\mu - 1 - \bar{A} + \bar{\alpha} - \alpha_2}{\bar{\gamma} + \gamma_2 - 1} = \sum_{\rho} b^{(1)}_{\gamma_2, \rho} \binom{\mu - \gamma_2 - \rho - 1 - \bar{A}}{\bar{\gamma} - 1}$$

and thus from (23) and (31)

$$(33) \quad D^{(\mu)}_{\gamma_2 \cdots \gamma_{\mu+\nu}} = \sum_{\rho} b^{(1)}_{\gamma_2, \rho} D^{(\mu-\gamma_2-\rho)}_{0\gamma_3 \cdots \gamma_{\mu+\nu}} .$$

Now if $j \geqq 4$ we separate all terms in $D^{(\mu-\gamma_2-\rho)}_{0\gamma_3 \cdots \gamma_{\mu+\nu}}$ which contain α_3 and apply (25) with $\nu = \alpha_3$, $n = \gamma_3$, and $m = 2$, obtaining

$$(34) \quad D^{(\mu)}_{\gamma_2 \cdots \gamma_{\mu+\nu}} = \sum_{\rho_2, \rho_3} b^{(1)}_{\gamma_2, \rho_2} b^{(2)}_{\gamma_3, \rho_3} D^{(\mu-\gamma_2-\gamma_3-\rho_2-\rho_3)}_{00\gamma_4 \cdots \gamma_{\mu+\nu}} .$$

We continue in this way and at each step the dependence of $D^{(\mu')}_{\gamma_2 \cdots \gamma_{\mu+\nu}}$ on a further $b^{(i-1)}_{\gamma_i, \rho_i}$ is expressed. Finally we consider $D^{(\mu-\gamma_2-\cdots-\gamma_{j-1}-\rho_2-\cdots-\rho_{j-1})}_{00\cdots0\gamma_j \cdots \gamma_{\mu+\nu}}$. Let $\mu' = \mu - \gamma_2 - \cdots - \gamma_{j-1} - \rho_2 - \cdots \rho_{j-1}$. Then by (23) and (29) (with $\alpha_j = \nu$, $n = \gamma_j$, $r = \mu' - \gamma_j$, $m = j - 1$) we have

$$(35) \quad D^{(\mu')}_{00\cdots0\gamma_j \cdots \gamma_{\mu+\nu}} = \sum (-1)^{\alpha_j} \binom{\gamma_j}{\alpha_j} \binom{\mu' - 1 - (j-1)\alpha_j}{\gamma_j - 1} = b^{(j-1)}_{\gamma_j, \mu-\gamma-\rho_2-\cdots}$$

Hence if $j \geqq 3$

$$(36) \quad D^{(\mu)}_{\gamma_2 \cdots \gamma_{\mu+\nu}} = \sum_{\rho_2, \ldots, \rho_{j-1}} b^{(1)}_{\gamma_2, \rho_2} b^{(2)}_{\gamma_3, \rho_3} \cdots b^{(j-2)}_{\gamma_{j-1}, \rho_{j-1}} b^{(j-1)}_{\gamma_j, \mu-\gamma-\rho_2-\cdots-\rho_{j-1}} .$$

Since $b^{(m)}_{0,k} = 0$ for $k > 0$ and $b^{(m)}_{0,k} = 1$ for $k = 0$, we may write (30) and (36) in the common form (valid for each admissible set of the γ_i's)

$$(37) \quad D^{(\mu)}_{\gamma_2 \cdots \gamma_{\mu+\nu}} = \sum b^{(1)}_{\gamma_2, \rho_2} b^{(2)}_{\gamma_3, \rho_3} \cdots b^{(\mu+\nu-1)}_{\gamma_{\mu+\nu}, \rho_{\mu+\nu}} ; \quad \rho_2 + \rho_3 + \cdots + \rho_{\mu+\nu} = \mu - \gamma$$

In particular we see by (19), (21), (22) and (37) that the $c_{\mu\nu}$ are *polynomials in* $a_2, a_3, \cdots$ *with non-negative integer coefficients.*

An elegant expression can be given to (37) in the following way. By the definition of the $b^{(m)}_{n,k}$ we have

$$\left(\frac{1 - x^{\lambda-1}}{1 - x} \right)^{\gamma_\lambda} = \sum_{\rho_\lambda=0}^{\infty} b^{(\lambda-1)}_{\gamma_\lambda, \rho_\lambda} x^{\rho_\lambda}$$

and so

$$\left[\prod_{\lambda=2}^{\mu+\nu} \left(\frac{1 - x^{\lambda-1}}{1 - x} \right)^{\gamma_\lambda} \right]_{\mu-\gamma} = \sum_{\rho_2, \ldots, \rho_{\mu+\nu}} b^{(1)}_{\gamma_2, \rho_2} b^{(2)}_{\gamma_3, \rho_3} \cdots b^{(\mu+\nu-1)}_{\gamma_{\mu+\nu}, \rho_{\mu+\nu}} .$$

Hence we have the following

THEOREM. *Let* $\gamma = \gamma_2 + \gamma_3 + \cdots + \gamma_{\mu+\nu}$, $\Gamma = 2\gamma_2 + 3\gamma_3 + \cdots + (\mu+\nu)\gamma_{\mu+\nu}$. *Then*

$$(38)\quad c_{\mu\nu} = \sum_{\Gamma=\mu+\nu} \frac{\mu(\gamma-1)!}{\gamma_2!\,\gamma_3!\cdots\gamma_{\mu+\nu}!} \left[\prod_{\lambda=2}^{\mu+\nu}\left(\frac{x-x^{\lambda}}{1-x}\right)^{\gamma_\lambda}\right]_{\mu} a_2^{\gamma_2} a_3^{\gamma_3}\cdots a_{\mu+\nu}^{\gamma_{\mu+\nu}},$$

where $[\cdots]_{\mu}$ *denotes the* μ*-th coefficient of* x *in the expansion of the expression in the bracket.*

Grunsky's law of symmetry, namely that $\nu c_{\mu\nu} = \mu c_{\nu\mu}$, may be derived immediately from (38). For

$$\nu C^{(\mu,\nu)}_{\gamma_2\cdots\gamma_{\mu+\nu}} = \mu\nu \frac{(\gamma-1)!}{\gamma_2!\cdots\gamma_{\mu+\nu}!} D^{(\mu)}_{\gamma_2\cdots\gamma_{\mu+\nu}}$$

and

$$\mu C^{(\nu,\mu)}_{\gamma_2\cdots\gamma_{\mu+\nu}} = \mu\nu \frac{(\gamma-1)!}{\gamma_2!\cdots\gamma_{\mu+\nu}!} D^{(\nu)}_{\gamma_2\cdots\gamma_{\mu+\nu}}.$$

Therefore, we have only to prove that

$$(39)\qquad \left[\prod_{\lambda=2}^{\mu+\nu}\left(\frac{1-x^{\lambda-1}}{1-x}\right)^{\gamma_\lambda}\right]_{\mu-\gamma} = \left[\prod_{\lambda=2}^{\mu+\nu}\left(\frac{1-x^{\lambda-1}}{1-x}\right)^{\gamma_\lambda}\right]_{\nu-\gamma}.$$

Now the expression

$$(40)\quad q(x) = \prod_{\lambda=2}^{\mu+\nu}\left(\frac{1-x^{\lambda-1}}{1-x}\right)^{\gamma_\lambda} = 1 + q_1x + \cdots + q_nx^n,\quad n = \sum_{\lambda=2}^{\mu+\nu}(\lambda-2)\gamma_\lambda$$

satisfies the equation $x^n q(1/x) = q(x)$ which yields $q_{n-s} = q_s$. Since $n = \Gamma - 2\gamma = (\mu-\gamma) + (\nu-\gamma)$ we thus have $q_{\mu-\gamma} = q_{\nu-\gamma}$, which is equivalent to (39).

NOTION DE PROXIMITÉ ET ESPACES À STRUCTURE SPHÉROÏDALE.*

By Lelio I. Gama.

1. Terminologie adoptée. *Espace accessible* = classe (H) de Fréchet.

Espace régulièrement accessible = espace accessible où à chaque point a correspond une suite évanescente de voisinages, c'est-à-dire une suite de voisinages

$$(1) \qquad V_1(a) \supset V_2(a) \supset \cdots \supset V_n(a) \supset \cdots$$

tels que tout $V(a)$ soit un sur-ensemble de $V_n(a)$ pour n suffisamment grand. Les voisinages (1) seront nommés voisinages *principaux* de a; $V_n(a)$ est le voisinage principal de rang n.

Espace strictement accessible = espace accessible où, a et b étant des points distincts, il y a un $V(a)$ et un $V(b)$ tels que $V(a) \cdot V(b) = 0$. C'est, donc, l'espace topologique de Hausdorff.

Nous dirons qu'un espace régulièrement accessible est *uniformément accessible*, si, pour tout voisinage V d'un point a, il en existe un autre $V' \subset V$ et un nombre naturel n' tels que

$$\alpha \in V' \cdot n > n' \rightarrow V_n(\alpha) \subset V.$$

2. Proximité de deux points. Nous dirons que deux points a et b d'un espace régulièrement accessible sont *proches* (l'un de l'autre), s'il existe un nombre naturel n tel que l'on ait

$$a \in V_n(b), \qquad b \in V_n(a).$$

La borne supérieure des n pour lesquels ces conditions se vérifient s'appellera *l'ordre de proximité*, ou, d'une façon abrégée, la *proximité* des points a et b; on la représentera par $\pi(a, b)$. Si a et b sont proches et $a \neq b$, $\pi(a, b)$ est un nombre naturel fini > 0. Si $a = b$, on a $\pi(a, b) = \infty$. Si a et b ne sont pas proches, nous poserons $\pi(a, b) = 0$. L'égalité $\pi(a, b) = n \neq 0$ entraîne, donc, toujours $a \in V_n(b)$, $b \in V_n(a)$.

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3. Application à la théorie générale des suites. Nous dirons que deux suites $\{a_n\}$, $\{b_n\}$ de points d'un espace régulièrement accessible sont *contiguës*, si, étant donné le nombre naturel N, il existe n' tel que $n > n' \rightarrow \pi(a_n, b_n) \geqq N$.

Dans un espace uniformément accessible deux suites contiguës ont le même ensemble-limite.[1] *En particulier, si l'une est convergente et a pour limite l, l'autre sera aussi convergente et aura pour limite l.*

Soient $\{a_n\}$ et $\{b_n\}$ contiguës dans un espace uniformément accessible. Posons d'abord $\lim a_n = l$. Prenons un voisinage V de l. Il y a un second voisinage $V' \subset V$ de l et un nombre naturel ν tels que tout voisinage d'ordre ν d'un point situé dans V' soit contenu dans V. Donc on peut déterminer k de façon que les voisinages d'ordre ν des points a_n $(n > k)$ soient tous contenus dans V. En vertu de la contiguïté de $\{a_n\}$ et $\{b_n\}$ on peut déterminer $n' > k$, tel que

$$n > n' \rightarrow \pi(a_n, b_n) \geqq \nu \rightarrow b_n \in V_\nu(a_n).$$

Mais $n > n' \rightarrow V_\nu(a_n) \subset V$, car $n' > k$, donc $n > n' \rightarrow b_n \in V$, et $\lim b_n = b$. Ainsi la convergence d'une des suites vers une limite l entraîne la convergence de l'autre vers la même limite. L'identité des ensembles-limites des deux suites en resultera aisément, car dans un espace régulièrement accessible, l'ensemble-limite d'une suite est l'ensemble des limites de ses sous-suites convergentes.

Nous dirons qu'une suite $\{a_n\}$ de points d'un espace régulièrement accessible est *convergente-C*, si, étant donné le nombre naturel $N > 0$, il existe n' tel que

$$p, q > n' \rightarrow \pi(a_p, a_q) \geqq N.$$

On y reconnaît une extension de la condition de Cauchy, ce qui justifie la dénomination adoptée.

Pour que $\{a_n\}$ soit convergente-C il faut et il suffit que deux sous-suites quelconques de $\{a_n\}$ soient contiguës.

Soient

$$a_{\nu(1)}, a_{\nu(2)}, \cdots, a_{\nu(n)}, \cdots$$
$$a_{\mu(1)}, a_{\mu(2)}, \cdots, a_{\mu(n)}, \cdots$$

[1] *Point-limite* de $\{a_n\}$ est tout point a tel que, pour tout voisinage V de a, la condition $a_n \in V$ se vérifie pour une infinité de valeurs de l'indice n. L'*ensemble-limite* de $\{a_n\}$ est l'ensemble de ses points-limites. Dans un espace accessible il est toujours fermé, et pour qu'il ne soit pas vide il suffit que le support de la suite (ensemble de ses points distincts) soit compact. Dans un espace régulièrement accessible l'ensemble-limite d'une suite est l'ensemble des limites des sous-suites convergentes.

deux sous-suites (infinies) de la suite convergente-C $\{a_n\}$. (On suppose que les indices ν et μ soient croissants). Le nombre naturel $N > 0$ étant donné, il existe n' tel que

$$(1) \qquad p, q > n' \rightarrow \pi(a_p, a_q) \geqq N.$$

Déterminons n_1 tel que, pour $n > n_1$, on ait à la fois $\nu(n) > n'$, $\mu(n) > n'$. Alors, en vertu de (1),

$$n > n_1 \rightarrow \pi[a_{\nu(n)}, a_{\mu(n)}] \geqq N,$$

ce qui prouve la contiguïté des deux sous-suites considerées. Pour prouver que la condition est suffisante, supposons que $\{a_n\}$ ne soit pas convergente-C, c'est-à-dire qu'il existe un nombre naturel $\nu > 0$ tel que, quel que soit n, on puisse toujours trouver des indices $p, q > n$, $p < q$, qui donnent

$$\pi(a_p, a_q) < \nu.$$

On pourra alors déterminer deux suites (infinies) de nombres naturels croissants, $\{p_n\}$ et $\{q_n\}$, $p_n < q_n$, pour lesquelles $\pi(a_{p_n}, a_{q_n}) < \nu$, et les deux sous-suites correspondantes, $\{a_{p_n}\}$, $\{b_{q_n}\}$ ne seront pas contiguës.

Si l'on convient d'entendre par ensemble *borné* dans un espace régulièrement accessible un ensemble dont tous les points appartiennent à la somme d'un nombre *fini* de voisinages principaux de points de cet espace, on pourra démontrer aisément que

Le support d'une suite convergente-C est un ensemble borné.

Car le nombre naturel $N > 0$ étant choisi arbitrairement, on peut trouver un autre nombre naturel $\nu > 0$ tel que $n > \nu \rightarrow \pi(a_\nu, a_n) \geqq N$, donc

$$n > \nu \rightarrow a_n \epsilon V_N(a_\nu),$$

ce qui montre que tous les points de $\{a_n\}$ sont contenus dans les voisinages d'ordre N des points $a_1, a_2, \cdots, a_\nu$.

On peut même affirmer, comme on le voit, que le support d'une suite convergente-C est *totalement borné* (dans un espace régulièrement accessible), c'est-à-dire qu'il est contenu dans un nombre fini de voisinages d'un rang aussi grand que l'on veut donné d'avance.

Dans un espace uniformément accessible, aucune suite convergente-C n'est discrépante.

En effet, si quelqu'une des sous-suites d'une suite convergente-C a une limite l, il en sera de même, par contiguïté, de toutes les autres, l'espace étant uniformément accessible. Il s'ensuit que l'ensemble-limite de la suite considérée ou bien est vide ou bien n'a qu'un seul élément. (Nous entendons ici

par suite *discrépante* une suite dont l'ensemble limite contient au moins deux éléments).

Dans un espace uniformément accessible, toute suite convergente-C, dont le support est compact, est convergente.

Car l'ensemble-limite d'une telle suite se réduit à un seul élément, l, et cela suffit pour que l soit limite de la suite, son support étant compact.

Dans un espace uniformément accessible, la convergence d'une sous-suite d'une suite convergente-C, suffit pour assurer et la convergence de la suite et la compacité de son support.

Soit $\{a_n\}$ la suite convergente-C dont la sous-suite

$$a_{\nu(1)}, \cdots, a_{\nu(n)}, \cdots \tag{1}$$

a pour limite l. Il suffit de prouver que le support A de $\{a_n\}$ est compact. Supposons, au contraire, que A contienne un sous-ensemble A_1 infini dont le dérivé soit nul. Soit

$$a_{\mu(1)}, \cdots, a_{\mu(n)}, \cdots \tag{2}$$

la sous-suite de $\{a_n\}$ dont le support est l'ensemble $A_1 - (l)$. Prenons un voisinage V_1 de l tel que

$$V_1[A_1 - (l)] = 0,$$

ce qui est possible car $A'_1 = 0$. Grâce à la condition que l'espace est uniformément accessible, il existe un autre voisinage $V_2 \subset V_1$ de l et un nombre naturel $m > 0$, tels que

$$\alpha \epsilon V_2 \rightarrow V_m(\alpha) \subset V_1,$$

donc

$$\alpha \epsilon V_2 \rightarrow V_m(\alpha)[A_1 - (l)] = 0.$$

Mais on peut déterminer k de façon que $n > k \rightarrow a_{\nu(n)} \epsilon V_2$, donc

$$n > k \rightarrow V_m(a_{\nu(n)})[A_1 - (l)] = 0.$$

Ceci montre que, pour $n > k$, le voisinage d'ordre m du point $a_{\nu(n)}$ de la suite (1) ne contient aucun point de la suite (2), et, par conséquent,

$$n > k \rightarrow \pi[a_{\nu(n)}, a_{\mu(n)}] < m,$$

ce qui est impossible, car $\{a_n\}$ étant convergente-C, les sous-suites (1) et (2) sont contiguës.

De ce qui précède on conclut que

Dans un espace uniformément accessible, la condition nécessaire et suffi-

sante pour qu'une suite convergente-C soit convergente est que son support soit compact.

La notion d'ensemble complet s'étendra aux espaces régulièrement accessibles (= classes (H) de Fréchet à caractère dénombrable). On y dira que l'ensemble E est complet, s'il est vide ou si toute suite convergente-C formée avec des points de E est convergente. En s'appuyant sur les propositions qu'on vient de démontrer, on verra tout de suite que:

Dans un espace uniformément accessible: 1° ***tout ensemble compact est complet; 2° pour qu'un ensemble E soit complet, il suffit que tout sous-ensemble borné de E soit compact; 3° la réunion d'un nombre fini d'ensembles complets est un ensemble complet.***

4. Ensembles clos. Considérons la borne supérieure P des proximités mutuelles des points d'un ensemble E, et appelons *ouverture intérieure* de E le nombre $\omega(E)$ défini par les conditions

$$\omega(E) = 1/P, \text{ si } 0 < P < \infty\,; \quad \omega(E) = 0, \text{ si } P = \infty\,; \quad \omega(E) = \infty, \text{ si } P = 0.$$

Cette notion s'applique à tout ensemble situé dans un espace régulièrement accessible. Nous dirons qu'un ensemble E d'un tel espace est *clos*, si $E = 0$, ou si tout sous-ensemble e de E, pour lequel $\omega(e) \neq 0$, est fini. Tout ensemble fini est clos. Dire qu'un ensemble infini E est clos équivant à dire que, pour tout sous-ensemble infini e de E, on a $\omega(e) = 0$. Dans un espace métrique dire que l'ensemble infini E est clos c'est dire que tout sous-ensemble infini e de E a son diamètre intérieur (= borne inférieure des distances mutuelles des points de e) égal à zéro.

Nous avons déjà convenu de dire qu'un ensemble est totalement borné quand on peut le renfermer dans un nombre fini de voisinages principaux d'un rang aussi élevé que l'on voudra.

Dans un espace régulièrement accessible, tout ensemble clos est totalement borné (la démonstration s'appuie sur l'axiome du choix).

Soit E un ensemble clos bien ordonné. Donnons-nous le nombre naturel $N > 0$. Supposant connu un sous-ensemble $E_n \neq 0$ de E, on peut définir, comme il suit, un sous-ensemble E_{n+1} de E_n: en désignant par a_n le premier élément de E_n, on appellera E_{n+1} l'ensemble des points a de E_n pour lesquels $\pi(a_n, a) < N$. E_{n+1} peut d'ailleurs être vide. En prenant pour E_1 l'ensemble E lui-même, on définit par là une suite d'ensembles décroissants

$$E = E_1 \supset E_2 \supset E_3 \supset \cdots,$$

dont les premiers éléments $a_1, a_2, a_3, \cdots$ vérifient la condition

(1) $$\pi(a_p, a_q) < N, \qquad (p < q).$$

Nous allons montrer que la suite $a_1, a_2, a_3, \cdots$ est finie. En vertu de (1), les points $a_1, a_2, \cdots$ sont distincts et leurs proximités mutuelles ont une borne supérieure $< N$. L'ensemble A de ces points a donc une ouverture intérieure $\omega(A) \neq 0$. Il en résulte que l'ensemble A, étant un sous-ensemble de l'ensemble clos E, est fini. La suite $\{a_n\}$ se réduit donc bien à un nombre fini de points

(2) $$a_1, a_2, \cdots, a_k,$$

ce qui revient à supposer $E_{k+1} = 0$. Si alors ξ est un point quelconque de E, il ne peut pas arriver qu'on ait à la fois

$$\pi(a_1, \xi) < N, \ \pi(a_2, \xi) < N, \cdots, \pi(a_k, \xi) < N,$$

puisque le système de ces inégalités entraînerait $\xi \in E_{k+1}$, contrairement à l'hypothèse $E_{k+1} = 0$. Il y a donc au moins un point de la suite (2), soit a_i, pour lequel $\pi(a_i, \xi) \geq N$, d'où $\xi \in V_N(a_i)$. Ainsi l'on voit que

$$E \subset V_N(a_1) \dotplus V_N(a_2) \dotplus \cdots \dotplus V_N(a_k),$$

c'est-à-dire que E est totalement borné.

5. Espaces à structure sphéroïdale.[2] On peut, moyennant la notion de proximité, définir une classe particulière d'espaces régulièrement accessibles. Nous dirons qu'un tel espace (c'est-à dire une classe (H) de Fréchet à caractère dénombrable) est un espace *à structure sphéroïdale,* si pour chaque nombre naturel $N > 0$, il en existe un autre ν, tel que, a et b étant des points, distincts ou non, de l'espace,

(1) $$[V_\nu(a) \cdot V_\nu(b) \neq 0] \cdot [\alpha, \beta \in V_\nu(a) + V_\nu(b)] \rightarrow \pi(\alpha, \beta) \geq N.$$

Si cette condition se vérifie pour l'indice ν, il en sera évidemment de même lorsqu'on y remplace ν par $\mu > \nu$. La plus petite valeur de l'indice ν pour laquelle (1) se vérifie est le *module de proximité N.*

Il s'ensuit de la définition que, dans un espace à structure sphéroïdale, à tout nombre naturel $N > 0$, il correspond un autre ν tel que, quel que soit le point a de l'espace,

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[2] A spheroidal space is, by (2), a space of uniform structure in the sense of A. Weil with the added restriction that the uniformity is (by **1** (1)) countable. Cf. A. Weil, "Sur les espaces à structure uniforme et sur la topologie générale," *Actualités scientifiques et industrielles,* no. 551 (1937). Weil's hypothesis U_{III}, p. 7, is essentially **5** (2) of the present paper and his hypotheses U_I and U_{II}, p. 7, are assured by **1** (1).

(2) $$\alpha, \beta \in V_\nu(a) \rightarrow \pi(\alpha, \beta) \geq N.$$

Nous appellerons *sphéroïde de centre a et d'ordre N*, en symboles: $\sigma(a, N)$, le premier voisinage $V_n(a)$ $(n = 1, 2, \cdots)$ pour lequel la condition (2) se trouve vérifiée.

Voici les propriétés fondamentales d'un espace à structure sphéroïdale.

a) $N < N' \rightarrow \sigma(a, N) \supset \sigma(a, N')$.

b) *Quel que soit N*, $\sigma(a, N) \subset V_N(a)$.

[Car $\alpha \in \sigma(a, N) \rightarrow \pi(a, \alpha) \geq N \rightarrow \alpha \in V_N(a)$].

c) *La suite* $\sigma(a, n)$, $(n = 1, 2, \cdots)$, *constitue une famille de voisinages de a équivalente à la famille donnée des voisinages de a.*

d) *Si* ν *est le module de proximité N et si a est un point quelconque de l'espace*

$$n \geq \nu \rightarrow \begin{cases} V_n(a) \subset \sigma(a, N), & \text{(I)} \\ \bar{V}_n(a) \subset V_N(a). & \text{(II)} \end{cases}$$

(I) est une conséquence immédiate des définitions de ν et de $\sigma(a, N)$. Quant à (II), il suffit de montrer que $\bar{V}_\nu(a) \subset V_N(a)$. Soit $\alpha \in \bar{V}_\nu(a)$; tout voisinage de α contient un point de $V_\nu(a)$, donc $V_\nu(a) \cdot V_\nu(\alpha) \neq 0$. Il s'ensuit, compte tenu de ce que ν est le module de proximité N, que $\pi(a, \alpha) \geq N$, d'où $\alpha \in V_N(a)$.

On voit, d'après II, que l'axiome de régularité (cf. Sierpiński, *Intr. to General Topology*, cap. v) est satisfait.

e) *Étant donné le nombre naturel* $N > 0$, *il existe* n' *tel que, pour tout point a de l'espace*,

$$b \in \bar{V}_n(a) \cdot n \geq n' \rightarrow \bar{V}_n(a) \subset V_N(b).$$

Soient ν le module de proximité N et n' le module de proximité ν. Alors, d'après la propriété antérieure, $n \geq n' \rightarrow \bar{V}_n(a) \subset V_\nu(a)$. Soient b un point fixe et β un point variable de $\bar{V}_n(a)$ $(n \geq n')$; b et β appartiennent à $V_\nu(a)$. Par suite, vu que ν est le module de proximité N, on a $\pi(b, \beta) \geq N$, d'où $\beta \in V_N(b)$.

f) *Tout espace à structure sphéroïdale est strictement accessible* [= espace topologique de Hausdorff (à caractère dénombrable)].

Soient, en effet, a et b des points distincts d'un espace à structure sphéroïdale; posons $\pi(a, b) = n_1$ et prenons un nombre naturel $n_2 > n_1$. Soit ν le module de proximité n_2. Les voisinages $V_\nu(a)$ et $V_\nu(b)$ sont disjoints, car $V_\nu(a) \cdot V_\nu(b) \neq 0$ entrainerait $\pi(a, b) \geq n_2 > n_1$, tandis que $\pi(a, b) = n_1$.

g) *Tout espace à structure sphéroïdale est uniformément accessible.*

Soit V un voisinage quelconque de a. Prenons le nombre naturel N assez grand pour que l'on ait $V_N(a) \subset V$, et soit ν le module de proximité N. Si $\alpha \in V_\nu(a)$ et $\beta \in V_n(\alpha)$, $n \geq \nu$, on aura $\pi(a, \beta) \geq N$, car $V_\nu(a) \cdot V_n(\alpha) \neq 0$ et ν est le module de proximité N. Donc $\beta \in V_N(a)$. Ainsi

$$\alpha \in V_\nu(a) \cdot n \geq \nu \rightarrow V_n(\alpha) \subset V_N(a) \subset V, \qquad \text{c. q. f. d.}$$

6. Métrisation des espaces régulièrement accessibles. THÉORÈME.[3] *Pour qu'un espace régulièrement accessible* [= classe (H) de Fréchet à caractère dénombrable] *soit un espace métrique, il faut et il suffit que ce soit un espace à structure sphéroïdale.*

La condition est nécessaire. Il suffit de prendre pour voisinages principaux des points de l'espace métrique les sphères (ouvertes) de rayon $1/n$ $(n = 1, 2, \cdots)$. On constatera que les limitations[4]

$$\pi(a, b) \geq n, \qquad \rho(a, b) < 1/n$$

sont équivalentes, et, dès lors, si α et β sont des points situés dans $V_{4N}(a) \dotplus V_{4N}(b)$, et pourvu que $V_{4N}(a) \cdot V_{4N}(b) \neq 0$, on aura $\rho(\alpha, \beta) < 1/N$, donc $\pi(\alpha, \beta) \geq N$.

La condition est suffisante. En supposant connu le nombre naturel $\nu_n > 0$, posons $\nu_{n+1} = \nu_n + \mu_n$, où l'on désigne par μ_n le module de proximité ν_n dans un espace à structure sphéroïdale donné. En nous donnant arbitrairement la valeur de ν_1, nous aurons par là défini une suite infinie $\{\nu_n\}$ de nombres naturels croissants, où le terme ν_{n+1} est plus grand que le module de proximité ν_n. Appelons F_n le système des voisinages de rang ν_n: F_n forme une couverture de l'espace envisagé. Nous allons montrer, d'après le théorème de Alexandroff et Urysohn (Cf. Fréchet, *Espaces Abstraits*, 1928, p. 220), que $\{F_n\}$ est une suite monotone complète. 1° Soient U'_{n+1}, U''_{n+1} deux éléments, distincts ou non, de F_{n+1} ayant un point p en commun. Soit α un point de U'_{n+1} ou de U''_{n+1}. Comme U'_{n+1} et U''_{n+1} sont des voisinages principaux de rang ν_{n+1} et que ν_{n+1} est plus grand que le module de proximité ν_n, on aura $\pi(p, \alpha) \geq \nu_n$, d'où $\alpha \in V_{\nu_n}(p)$. Par suite U'_{n+1} et U''_{n+1} sont tous les deux contenus dans l'élément $V_{\nu_n}(p)$ de F_n. La suite $\{F_n\}$ est donc bien monotone. 2° Soient ξ un point quelconque de l'espace et $\{U_n\}$ une suite telle que $U_n \in F_n$, $\xi \in U_n$. Il s'agit de montrer que $\{U_n\}$ constitue une famille de voisinages de ξ équivalente à la famille donnée des voisinages $V(\xi)$ de ξ. Que tout U_n contient un $V(\xi)$ découle immédiatement de ce qu'on a affaire à une classe (H) de Fréchet. Réciproquement, soit $V(\xi)$ un voisinage donné de ξ. Prenons N de façon que

[3] Cf. A. Weil, *loc. cit.*[1] and J. W. Tukey, *Convergence and Uniformity in Topology*, Princeton (1940), p. 61, Theorem 6. 1.

[4] $\rho(a, b)$ = distance des points a et b.

$V_N(\xi) \subset V(\xi)$. Par la propriété (e) des espaces à structure sphéroïdale (4), il existe n' tel que, quel que soit le point a de l'espace,

$$\xi \in V_n(a) \cdot n \geq n' \rightarrow V_n(\alpha) \subset V_N(\xi) \subset V(\xi).$$

Comme U_n est un voisinage de rang ν_n d'un certain point a de l'espace, il suffit de prendre $\nu_n \geq n'$ pour que l'inclusion $U_n \subset V(\xi)$ se trouve vérifiée.

7. Rôle de la notion de proximité dans les espaces métriques. D'après le théorème qui précède, il est intéressant d'examiner le rôle que joue la notion de proximité dans les démonstrations des propriétés des espaces métriques, en cherchant à retrouver ces propriétés directement dans un espace à structure sphéroïdale. On y trouvera, peut-être, une méthode assez simple d'étudier les propriétés des espaces métriques sans faire appel à la notion de distance. Nous nous bornerons, dans ce numéro, à deux théorèmes classiques.

I. *La condition de normalité* (au sens strict). *Si A et B sont des ensembles séparés* ($A\bar{B} \dotplus \bar{A}B = 0$), *on peut déterminer deux ensembles ouverts P, Q, tels que $A \subset P$, $B \subset Q$, $PQ = 0$.*

À chaque point a de A associons le premier voisinage principal, $V_M(a)$, tel que $V_M(a) \cdot B = 0$, et, en appelant μ le module de proximité M, posons

$$P = \Sigma V_\mu(a),$$

la sommation s'étendant à tous les points a de A (μ varie, en général, avec le point a considéré). De même, associons à chaque point b de B le premier voisinage principal, $V_N(b)$, tel que $V_N(b) \cdot A = 0$, et, désignant par ν le module de proximité N, posons

$$Q = \Sigma V_\nu(b).$$

Il suffit de montrer que $PQ = 0$. Supposons, au contraire, que $k \in PQ$. Le point k appartiendra à un couple d'éléments $V_{\mu'}(a') \in P$ et $V_{\nu'}(b') \in Q$, et, si $V_{M'}(a')$, $V_{N'}(b')$ sont les voisinages associés aux points a', b', les indices μ', ν' seront, respectivement, les modules de proximité M' et de proximité N' Soit $\nu' \geq \mu'$. Alors, comme il s'agit d'un espace à structure sphéroïdale, on aura, pour deux points quelconques α, β pris dans $V_{\mu'}(a') \dotplus V_{\nu'}(b')$, $\pi(\alpha, \beta) \geq M'$. On a, en particulier, $\pi(a', b') \geq M'$, et, par suite, $b' \in V_{M'}(a')$, ce qui est impossible, d'après la définition même de $V_{M'}(a')$.

II. *Le théorème de Borel-Lebesgue.* Pour démontrer ce théorème dans un espace à structure sphéroïdale, on peut employer le mode de raisonnement utilisé par M. Fréchet dans sa démonstration directe du théorème pour le cas

des espaces métriques,[5] en s'appuyant aussi sur l'axiome du choix. Soit K l'ensemble compact et fermé, couvert par la famille infinie ΣA, de sorte que $K \subset \Sigma I$, I étant l'intérieur de A. Nous considérons K comme un ensemble bien ordonné. Associons à chaque point a de K le premier voisinage principal de a, $U(a)$, ayant la propriété d'être contenu dans un I. Il suffit de prouver que K peut être couvert par une collection finie des voisinages $U(a)$. Supposons que cela n'ait pas lieu. On peut alors, en partant du premier élément a_1 de K, et en utilisant le bon ordre de K, définir une suite infinie $\{a_n\}$ de points de K, de telle façon que a_n $(n > 1)$ ne soit contenu dans aucun des voisinages $U(a_1), \cdots, U(a_{n-1})$, associés, comme on l'a expliqué, aux points $a_1, \cdots, a_{n-1}$. Les a_n sont évidemment des points distincts. L'ensemble A des a_n étant compact (puisque $A \subset K$), on peut extraire de $\{a_n\}$ une sous-suite $\{\alpha_n\}$ convergente vers un point ξ du dérivé A'; ce point ξ appartient d'ailleurs à K (puisque K est fermé). Envisageons le voisinage $U(\xi)$, associé au point ξ. En vertu de l'uniforme accessibilité d'un espace à structure sphéroïdale, il existe un voisinage de ξ, $W \subset U(\xi)$, et un nombre naturel n', tels que

$$\alpha \,\epsilon\, W \cdot n > n' \rightarrow V_n(\alpha) \subset U(\xi).$$

Soit donc $N > n'$ un nombre naturel assez grand pour que l'on ait $\sigma(\xi, N) \subset W$, et considérons deux points α_i, α_{i+1} de la suite $\{\alpha_n\}$ contenus dans $\sigma(\xi, N)$, ce qui est possible, car $\lim \alpha_n = \xi$. On aura $\pi(\alpha_i, \alpha_{i+1}) \geq N$, d'où

$$(1) \qquad \alpha_{i+1} \,\epsilon\, V_N(\alpha_i).$$

Mais, d'autre part, puisque $N > n'$, $V_N(\alpha_i) \subset U(\xi)$, et, par suite, $V_N(\alpha_i)$ est contenu dans un I; il en résulte que $V_N(\alpha_i) \subset U(\alpha_i)$, puisque $U(\alpha_i)$ est le premier voisinage principal de α_i ayant la propriété d'être contenu dans un I. Comme, d'après la définition même de la suite $\{a_n\}$, α_{i+1} n'est pas contenu dans $U(\alpha_i)$, on aura

$$(2) \qquad \alpha_{i+1} \,\overline{\epsilon}\, V_N(\alpha_i).$$

Les conclusions (1) et (2) sont contradictoires.

III. *Remarque sur les ensembles totalement bornés.* Il nous sera utile d'avoir observé que, si E est un ensemble totalement borné dans un espace à structure sphéroïdale, il existe, pour chaque naturel $n > 0$, une collection finie de points de E, $\xi_1, \cdots, \xi_k$, tels que

$$E \subset V_n(\xi_1) \dotplus V_n(\xi_2) \dotplus \cdots \dotplus V_n(\xi_k).$$

[5] *American Journal of Mathematics*, vol. 50 (1928), p. 70.

Étant donné N, on peut, en effet, déterminer ν tel que, pour tout point a de l'espace et pour tout point $b \in \bar{V}_\nu(a)$, on ait (**5**, e)

$$(1) \qquad \bar{V}_\nu(a) \subset V_N(b).$$

D'autre part, puisque l'ensemble E est totalement borné, il admet une inclusion de la forme

$$E \subset V_\nu(a_1) \dot{+} V_\nu(a_2) \dot{+} \cdots \dot{+} V_\nu(a_k),$$

avec $EV_\nu(a_s) \neq 0$ $(s = 1, 2, \cdots, k)$. Alors, si $\xi_s \in EV_\nu(a_s)$, on aura, d'après (1),

$$V_\nu(a_s) \subset V_N(\xi_s), \qquad (s = 1, 2, \cdots, k)$$

donc

$$E \subset V_N(\xi_1) \dot{+} V_N(\xi_2) \dot{+} \cdots \dot{+} V_N(\xi_k).$$

8. Suites évanescentes d'ensembles.[6] Nous allons étudier les propriétés des suites évanescentes d'ensembles, définies dans un espace métrisable, en utilisant encore la notion de proximité au lieu de celle de distance, c'est-à-dire en prenant pour base un espace à structure sphéroïdale. Nous appellerons *suite évanescente d'ensembles* (dans un espace régulièrement accessible) toute suite $\{E_n\}$ d'ensembles satisfaisant aux deux conditions suivantes:

1° quels que soient les indices p, q, $\bar{E}_p\bar{E}_q \neq 0$;

2° à chaque E_n correspond un point a_n de $\bar{E}_n$ ayant cette propriété: étant donné le nombre naturel $\nu > 0$, il existe n' tel que $n > n' \rightarrow E_n \subset \bar{V}_\nu(a_n)$.

Le point a_n sera nommé le *centre* de E_n. Si une des suites $\{E_n\}$, $\{\bar{E}_n\}$ est évanescente, il en est de même de l'autre.

a) *Dans un espace à structure sphéroïdale, si* $\{E_n\}$ *est une suite évanescente et si* a_n *est le centre de* E_n, *alors étant donné le nombre naturel* $\nu > 0$, *il existe* n' *tel que* $n > n' \rightarrow \bar{E}_n \subset \sigma(a_n, \nu)$, *donc aussi* (**5**, b) $\bar{E}_n \subset V_\nu(a_n)$.

Grâce à la propriété d) (n. 5), on peut déterminer m de façon qu'on ait, quel que soit le point a de l'espace, $V_m(a) \subset \sigma(a, \nu)$, et, en suite, déterminer μ de façon que $\bar{V}_\mu(a) \subset V_m(a)$. On aura alors, quel que soit a, $\bar{V}_\mu(a) \subset \sigma(a, \nu)$. Or, d'après la définition d'une suite évanescente, on a $\bar{E}_n \subset \bar{V}_\mu(a_n)$ dès que l'indice n soit suffisamment grand.

b) *Si* $\{E_n\}$ *est évanescente dans un espace à structure sphéroïdale,* 1° *toute suite* $\{\xi_n\}$, *où* $\xi_n \in \bar{E}_n$, *est convergente-C*; 2° *si l'on prend dans chaque* $\bar{E}_n$ *un couple de points* ξ_n, η_n, *les suites* $\{\xi_n\}$, $\{\eta_n\}$ *seront contiguës.*

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[6] Cf. G. Birkhoff, "Moore-Smith convergence in general topology," *Annals of Mathematics*, vol. 38 (1937), pp. 39-60.

1° Soit donné le nombre naturel $N > 0$, et appelons ν le module de proximité N. Il existe n' tel que $n > n' \rightarrow \bar{E}_n \subset V_\nu(a_n)$; donc si p, q sont des nombres naturels $> n'$, on aura $V_\nu(a_p) \cdot V_\nu(a_q) \neq 0$, car, par hypothèse, $\bar{E}_p \cdot \bar{E}_q \neq 0$. Puisque ν est le module de proximité N, on en conclut que $p, q > n' \rightarrow \pi(\xi_p, \xi_q) \geqq N$, c. q. f. d.

2° D'après le théorème a), le nombre naturel N étant donné, les points ξ_n, η_n appartiendront à $\sigma(a_n, N)$, dès que n soit plus grand qu'une certaine valeur n'; donc, $n > n' \rightarrow \pi(\xi_n, \eta_n) \geqq N$.

Disons que λ est un *point-limite* de la suite évanescente $\{E_n\}$, lorsque, pour tout voisinage V de λ, il existe n' tel que $n > n' \rightarrow E_n \subset V$. Dans un espace strictement accessible une suite évanescente d'ensembles ne peut admettre qu'un seul point-limite.

c) Dans tout espace régulièrement accessible, si le point λ est point-limite de $\{\bar{E}_n\}$, il le sera évidemment aussi de $\{E_n\}$. *La réciproque est vraie, pourvu que l'espace soit à structure sphéroïdale.*

Soient, en effet, λ le point-limite de $\{E_n\}$, et V un voisinage de λ. Prenons un voisinage principal $V_\nu(\lambda) \subset V$. On peut déterminer (**5**, d) un $V_m(\lambda)$ tel que $\bar{V}_m(\lambda) \subset V_\nu(\lambda)$, et, par suite, l'inclusion $E_n \subset V_m(\lambda)$ entraîne $\bar{E}_n \subset V_\nu(\lambda) \subset V$, c. q. f. d.

On en conclut que, dans un espace à structure sphéroïdale, si λ est le point-limite de $\{E_n\}$, et si $\xi_n \in \bar{E}_n$, on aura $\lim \xi_n = \lambda$. Réciproquement,

Soit $\{E_n\}$ évanescente dans un espace à structure sphéroïdale. Si $\lambda = \lim \xi_n$, ($\xi_n \in \bar{E}_n$), alors λ sera le point-limite de $\{E_n\}$ et de $\{\bar{E}_n\}$.

Soit V un voisinage de λ. Grâce à l'uniforme accessibilité de l'espace, on peut determiner un voisinage de λ, $V_1 \subset V$, et un nombre naturel m, tels que

$$\alpha \in V_1 \rightarrow V_m(\alpha) \subset V.$$

La suite $\{a_n\}$ des centres des E_n converge vers λ, car elle est contiguë à $\{\xi_n\}$; on peut donc déterminer n' tel que $n > n' \rightarrow a_n \in V_1$, donc

$$n > n' \rightarrow V_m(a_n) \subset V.$$

D'autre part, il existe n'' tel que $n > n'' \rightarrow \bar{E}_n \subset V_m(a_n)$. On en conclut que, si N est un nombre naturel plus grand que n' et n'', $n > N \rightarrow \bar{E}_n \subset V$.

d) *Dans un espace à structure sphéroïdale, si λ est le point-limite d'une suite évanescente $\{E_n\}$, on a $\Pi \bar{E}_n = (\lambda)$.*

Soit V un voisinage de λ. D'après c), on peut prendre un E_m tel que

$\bar{E}_m \subset V$. Comme, en vertu de la condition 1° d'évanescence, $\bar{E}_n\bar{E}_m \neq 0$ quel que soit n, on aura, pour chaque valeur de n, $\bar{E}_n V \neq 0$, donc $\lambda \in \bar{E}_n$ $(n = 1, 2, \cdots)$, c'est-à-dire $\lambda \in \Pi\bar{E}_n$. Soient $\lambda_1 \neq \lambda$ et V un voisinage de λ ne contenant pas λ_1. On peut trouver un $\bar{E}_m \subset V$, donc $\lambda_1 \bar{\in} \bar{E}_m$, et, par suite, $\lambda_1 \bar{\in} \Pi\bar{E}_n$.

e) Si $\{\bar{E}_n\}$ est évanescente dans un espace à structure sphéroïdale et si $\Pi\bar{E}_n \neq 0$, alors $\Pi\bar{E}_n$ se réduit à un point unique, qui est le point-limite de $\{E_n\}$ et de $\{\bar{E}_n\}$.

Soient $\lambda \in \Pi\bar{E}_n$ et V un voisinage de λ. Prenons $V_N(\lambda) \subset V$. Par la propriété a), il existe n' tel que $n > n'$ $\bar{E}_n \subset \sigma(a_n, N)$, a_n désignant le centre de E_n. Pour tout point ξ de $\bar{E}_n$ $(n > n')$ on aura donc $\pi(\lambda, \xi) \geq N$, d'où $\xi \in V_N(\lambda) \subset V$. On voit que $n > n' \rightarrow \bar{E}_n \subset V$, c'est-à dire que λ est *le* point-limite de $\bar{E}_n$.

f) Si $\{E_n\}$ est évanescente dans un espace à structure sphéroïdale, et si a_n est le centre de E_n, alors, pour tout nombre naturel donné $m > 0$, il existe N tel que $n \geq N \rightarrow \bar{E}_n \subset V_m(a_N)$.

Soit μ le module de proximité m. Il existe n' tel que (propriété a)

$$(1) \qquad n > n' \rightarrow \bar{E}_n \subset V_\mu(a_n),$$

et, puisque $\{a_n\}$ est convergente-C, on peut, d'autre part, déterminer $N > n'$ tel que $n \geq N \rightarrow \pi(a_N, a_n) \geq \mu$, donc

$$(2) \qquad n \geq N \rightarrow a_N \in V_\mu(a_n).$$

Soit alors $\xi \in \bar{E}_n$ $(n \geq N)$; il résulte de (1) et (2) que ξ et a_N sont tous les deux contenus dans $V_\mu(a_n)$. Donc, vu que μ est le module de proximité m, $\pi(a_N, \xi) \geq m$, et $\xi \in V_m(a_N)$. Ainsi, $n \geq N \rightarrow \bar{E}_n \subset V_m(a_N)$.

9. Ensembles cantoriens. À la notion de suite évanescente d'ensembles se rattache celle que nous allons envisager d'ensemble cantorien. Nous dirons que E est un ensemble *cantorien* (dans un espace régulièrement accessible), si $E = 0$ ou si, pour chaque suite évanescente

$$(1) \qquad V_{\nu(1)}(a_1), V_{\nu(2)}, \cdots, V_{\nu(n)}(a_n), \cdots$$

constituée de voisinages principaux de points a_n, distincts ou non, de l'ensemble donné E, il existe un point ω, et un seul, tel que

$$\Pi\bar{V}_{\nu(n)}(a_n) = (\omega).$$

Dans un espace à structure sphéroïdale, un ensemble cantorien non vide peut être défini par cette condition : toute suite telle que (1) doit admettre un point-

limite ω (c'est ce qui résulte des théorèmes d) et e) du n. précédent). Nous allons montrer que

Dans un espace à structure sphéroïdale, il y a identité entre les notions d'ensemble complet et d'ensemble cantorien.

Soit $E \neq 0$ un ensemble complet. Considérons une suite évanescente $\{V(a_n)\}$, dont chaque terme est un voisinage principal d'un point a_n $(n = 1, 2, \cdots)$ de E. $\{a_n\}$ est une suite convergente-C (**8**, b), qui a une limite ω, puisque E est complet. Donc (**8**, c) la suite $\{V(a_n)\}$ a pour point-limite ω, et, par suite, E est un ensemble cantorien.

Soit $E \neq 0$ un ensemble cantorien. Considérons une suite convergente-C, $\{a_n\}$, dont les termes sont des points de E. Soient $n(1)$ le plus petit nombre naturel pour lequel

$$n > n(1) \rightarrow \pi[a_{n(1)}, a_n] \geqq 1;$$

$n(2)$ le plus petit nombre naturel $> n(1)$, pour lequel

$$n > n(2) \rightarrow \pi[a_{n(2)}, a_n] \geqq 2,$$

et ainsi de suite. Il est manifeste qu'on définit par cette norme une suite (infinie) de nombres naturels croissants $n(1), n(2), \cdots, n(\nu), \cdots$ ayant cette propriété:

$$(1) \qquad n > n(\nu) \rightarrow \pi[a_{n(\nu)}, a_n] \geqq \nu.$$

Envisageons la suite suivante de voisinages principaux des points $a_{n(\nu)}$:

$$(2) \qquad V_1[a_{n(1)}], V_2[a_{n(2)}], \cdots, V_\nu[a_{n(\nu)}], \cdots.$$

Si μ est un nombre naturel $> \nu$, on a $n(\mu) > n(\nu)$, et, d'après (1), $\pi[a_{n(\nu)}, a_{n(\mu)}] \geqq \nu$, donc

$$\mu > \nu \rightarrow a_{n(\mu)} \in V_\nu[a_{n(\nu)}].$$

Ceci montre que chaque voisinage de la suite (2) contient les « centres » $a_{n(\nu)}$ de tous les voisinages suivants, et, par conséquent, cette suite remplit la condition 1° d'évanescence (**8**); elle satisfait en outre à la condition 2°, car elle est formée de voisinages principaux dont les rangs ν sont croissants. Et alors, puisque E est un ensemble cantorien, la suite (2) possède un point-limite ω, et on aura

$$\lim_{\nu=\infty} a_{n(\nu)} = \omega.$$

Mais $\{a_{n(\nu)}\}$, $\nu = 1, 2, \cdots$, est une sous-suite de la suite convergente-C $\{a_n\}$; il s'ensuit (**3**) que $\omega = \lim a_n$. Donc E est un ensemble complet.

10. Propriétés transitives. Nous dirons qu'une propriété P d'ensemble est *transitive*, si, pour tout ensemble non vide E jouissant de cette propriété, la condition suivante est remplie:

a) Pour toute décomposition $E = E_1 \dot{+} E_2$, l'une au moins des parties E_1, E_2 possède la propriété P.

Si l'on ajoute à cette condition la restriction suivante:

b) si le sous-ensemble e_1 de E ne jouit pas de la propriété P, il en sera de même de tout sous-ensemble (non vide) de e_1,
alors la propriété P deviendra ce que nous avons appelé une propriété *héréditaire*.[7]

Si $E \subset E_1 \dot{+} E_2 \dot{+} \cdots \dot{+} E_m$ $(E \neq 0)$, toute propriété transitive ou héréditaire de E se transmet à l'un au moins des produits $EE_1, EE_2, \cdots, EE_m$ qui ne soient pas vides.

Nous dirons qu'une propriété P d'un ensemble E $(\neq 0)$ est *localisable* en un point a, si tout voisinage de a contient un sous-ensemble $e \neq 0$ de E possédant la propriété P. [Cette définition de la localisation d'une propriété est un peu moins restrictive que celle que nous avons adopté dans l'article cité].

Dans un espace à structure sphéroïdale, toute propriété transitive d'un ensemble totalement borné et complet E $(\neq 0)$ est localisable en un point au moins de la fermeture de E.[8]

[Avec la définition plus restrictive de la localisation d'une propriété, ce théorème est valable pour les propriétés héréditaires. C'est là le point de vue où nous nous sommes placés dans l'article déjà cité].

Pour le démontrer, soit $E \neq 0$ un ensemble totalement borné et complet (dans un espace à structure sphéroïdale) jouissant d'une propriété transitive P. Nous considérons E comme un ensemble bien-ordonné. E étant totalement borné, il existe, pour chaque nombre naturel $n > 0$, des collections finies formées de voisinages de rang n de points appartenant à E, chaque collection couvrant E. D'après le principe du choix, on peut donc, à chaque nombre naturel $n > 0$, faire correspondre une telle collection, soit K_n. Les éléments de K_n (voisinages de rang n de points de E) seront censés ordonnés dans l'ordre même de ces points.

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[7] "Sur quelques points de la théorie des ensembles abstraits et la notion d'accumulatif," *Anais da Academia Brasileira de Ciencias*, vol. 12 (1940). Dans la définition donnée dans cet article nous avions employé l'expression « l'ensemble E » dans un sens générique, en y laissant donc sous-entendu que les deux conditions a) et b) devraient être remplies pour *tout* ensemble E ayant la propriété envisagée.

[8] La démonstration s'appuie sur l'axiome du choix.

Puisque $E \subset K_1$, il existe un premier voisinage U_1 dans la collection K_1, tel que

$$EU_1 \neq 0, \qquad EU_1 \in (P),$$

(P) désignant la classe des ensembles qui jouissent de la propriété considérée P. On peut alors définir une suite infinie $U_1, \cdots, U_n, \cdots, (U_n \in K_n)$, ayant cette propriété: quel que soit n,

$$(1) \qquad EU_1U_2 \cdots U_n \neq 0,$$

$$(2) \qquad EU_1U_2 \cdots U_n \in (P).$$

Car, une fois determinée une suite finie $U_1, \cdots, U_m$ remplissant les conditions (1), (2) pour $n = 1, 2, \cdots, m$, le produit $EU_1 \cdots U_m$, qui est un sous-ensemble non vide de E, est contenu dans K_{m+1}, et, puisqu'il a la propriété P, on trouvera dans la collection K_{m+1} un premier voisinage, U_{m+1}, tel que

$$0 \neq EU_1 \cdots U_mU_{m+1} \in (P).$$

Il est maintenant aisé de voir que la suite $\{U_n\}$ ainsi définie est une suite évanescente (**8**); car, d'abord, p et q étant des nombres naturels > 0, on a, en vertu de la condition (1), $U_pU_q \neq 0$, ce qui montre que la condition 1° d'évanescence est remplie. D'autre part, soit $\nu > 0$ un nombre naturel, et désignons par a_n le point de E dont U_n est un voisinage. Il est clair que

$$n > \nu \rightarrow U_n(a_n) \subset V_\nu(a_n) \subset \bar{V}_\nu(a_n),$$

ce qui fait voir que la condition 2° d'évanescence est aussi remplie.

E est un ensemble cantorien, parce qu'il est complet dans un espace à structure sphéroïdale (**9**). Il s'ensuit que $\{U_n\}$ admet un point-limite ω. Soit V un voisinage quelconque de ω: il y a un $U_m \subset V$, et, par suite, $EU_1 \cdots U_m \subset V$. Mais l'ensemble $EU_1 \cdots U_m$ possède la propriété P d'après la condition (2). La propriété P est donc bien localisable au point ω. L'on voit en même temps que $\omega \in \bar{E}$, comme on devait s'y attendre. Le théorème est donc démontré.

En voici une application.

Dans un espace à structure sphéroïdale, la condition nécessaire et suffisante pour qu'un ensemble soit compact, est que cet ensemble soit totalement borné et complet.

Soit, en effet, E un sous-ensemble infini d'un ensemble totalement borné et complet (dans un espace à structure sphéroïdale). E sera lui-même un

ensemble totalement borné et complet. Or, la propriété pour un ensemble d'être infini est transitive. Il y a donc un point ω dont tout voisinage contient une infinité de points de E. Mais alors $\omega \epsilon E'$, donc $E' \neq 0$. La condition est donc suffisante. La condition est nécessaire, parce que, d'abord, dans un espace uniformément accessible quelconque, un ensemble compact est toujours complet, et, d'autre part, il est facile de voir que, dans un espace à structure sphéroïdale, un ensemble compact est toujours clos (et, par suite, totalement borné, **4**). En effet, si $a \epsilon E'$, et si $N > 0$ est un nombre naturel quelconque, il y a un point $b \neq a$ de E contenu dans le sphéroïde $\sigma(a, N)$ (**5**, c) ; par suite $\pi(a, b) \geq N$ et $\omega(E) = 0$ (**4**).

11. Note sur le théorème de Borel-Lebesgue. D'après cela, le théorème du numéro précédent sur la localisation des propriétés transitives peut être énoncé sous cette forme :

Dans un espace à structure sphéroïdale, toute propriété transitive d'un ensemble compact E ($\neq 0$) est localisable en un point au moins de la fermeture de E.

Sous cette forme, cette proposition peut être établie bien aisément en s'appuyant sur le théorème de Borel-Lebesgue. Mais il est intéressant de remarquer [9] que ce dernier théorème n'est lui-même qu'une forme particulière de la proposition qu'on vient d'énoncer, c'est-à-dire que la propriété de Borel-Lebesgue découle, dans les espaces à structure sphéroïdale, d'un processus de localisation d'une propriété transitive (ou héréditaire, lorsqu'on adopte la définition plus restrictive du concept « localisable »).[9] Voici comment on peut le montrer.

Soit E un ensemble compact et fermé couvert par une famille infinie F d'ensembles ϕ, dans un espace à structure sphéroïdale. Il est manifeste que la propriété P pour un ensemble de n'être pas couvert par une collection finie des ensembles ϕ est transitive. Supposons, alors, que l'ensemble donné E ait cette propriété P. Soit $\omega \epsilon \bar{E} = E$ un point de localisation de la propriété P. Le point ω (parce qu'il appartient à E) est dans l'intérieur I d'un ensemble ϕ_1 de la famille de couverture F. Soit V un voisinage de ω tel que $V \subset I$. Puisque la propriété P est localisable au point ω, V contient un sous-ensemble e de E qui n'est couvert par aucune collection finie des ensembles ϕ, ce qui est en contradiction avec l'inclusion $e \subset \phi_1$ ($e \subset V \subset I \subset \phi_1$).

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[9] Voir l'article déjà cité *Sur quelques points* etc.

DUALITY THEOREMS FOR GENERALIZED MANIFOLDS.*

By Edward G. Begle.[1]

In a previous paper [2] we gave a new definition of a generalized manifold.[3] By combining cohomology theory and some of our results on homology local connectedness, we were able to give a short proof of the Poincaré Duality Theorem for these manifolds. The object of the present paper is to show that the same methods can be used to give equally simple proofs of several other duality theorems, including, in particular, the Alexander Duality Theorem, for generalized manifolds.

The proofs require a discussion of open generalized manifolds, i.e., locally compact spaces with the local homology and cohomology properties of Euclidean spaces. Accordingly, the first four sections below are devoted to the required definitions and the changes necessitated in passing from the compact to the locally compact case. The Poincaré Duality Theorem is proved in Section **5** and is used in Section **6** in proving other duality theorems. We conclude in Section **7** by showing that the algebraic isomorphisms established in the Alexander Duality Theorem also have a useful geometric property.

1. Homology and cohomology theory of locally compact spaces. In addition to the ordinary Čech homology and cohomology groups of a topological space, there are two other pairs of groups which are necessary for the treatment of open generalized manifolds. In the following, the coefficient group is assumed to be an arbitrary but fixed field.

Let $\mathfrak{S}$ be a locally compact [4] space and let $\{L_\lambda\}$ be the collection of all compact subsets of $\mathfrak{S}$. We partially order the collection $\{L_\lambda\}$ by: $L_\lambda < L_\mu$ if, and only if, $L_\lambda \supset L_\mu$. Since the sum of two compact sets is compact, it is

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* Received March 13, 1943.

[1] Most of the results of this paper were obtained while the author was a National Research Fellow.

[2] "Locally connected spaces and generalized manifolds," *American Journal of Mathematics,* vol. 64 (1942), pp. 553-574. This paper will hereafter be referred to as [B]. Numbers in square brackets refer to the bibliography at the end of [B]. A knowledge of [B] is necessary for the understanding of the present paper.

[3] Our generalized manifold includes and is very similar to Čech's [5]. If the space in question is compact metric, both types are the same as Wilder's [20].

[4] Compact = bicompact. All spaces considered here are understood to be normal.

clear that if L_μ and L_ν are arbitrary then $L_\lambda = L_\mu \cup L_\nu$ is such that $L_\lambda < L_\mu$. $L_\lambda < L_\nu$, i. e., $\{L_\lambda\}$ is a directed set [17].

Let z_λ^p denote the group of p-dimensional Čech cycles of $\mathfrak{S}$ which are carried [5] by L_λ, and let f_λ^p denote the subgroup of z_λ^p consisting of those cycles which bound on L_λ. If $L_\lambda < L_\mu$, the identity mapping is a homomorphism $\sigma^*{}_\lambda{}^\mu$ of z_μ^p into z_λ^p. Hence $\{z_\lambda^p, \sigma^*{}_\lambda{}^\mu\}$ and $\{f_\lambda^p, \sigma^*{}_\lambda{}^\mu\}$ are direct systems of groups [see 15, p. 57]. We denote their limit groups by $z^p(\mathfrak{S})$ and $f^p(\mathfrak{S})$. The quotient group, $h^p(\mathfrak{S})$, is called the p-dimensional compact homology group of $\mathfrak{S}$. The elements of $z^p(\mathfrak{S})$ are called compact cycles.

Note that since $\{z_\lambda^p, \sigma^*{}_\lambda{}^\mu\}$ is a direct system, a compact cycle of $\mathfrak{S}$ is determined, up to homology, by any one of its coordinates.

Next let $Z_p{}^\lambda$ be the group of p-dimensional Čech cocycles of $\mathfrak{S}$ mod $\mathfrak{S} - L_\lambda$ and let $F_p{}^\lambda$ be the subgroup of those cocycles which are ~ 0 mod $\mathfrak{S} - L_\lambda$. The identity mapping is a homomorphism $\sigma_\lambda{}^\mu$ of $Z_p{}^\lambda$ into $Z_p{}^\mu$. Hence $\{Z_p{}^\lambda, \sigma_\lambda{}^\mu\}$ and $\{F_p{}^\lambda, \sigma_\lambda{}^\mu\}$ are inverse systems of groups. The limit groups are denoted by $Z_p(\mathfrak{S})$ and $F_p(\mathfrak{S})$ and their quotient group by $H_p(\mathfrak{S})$. The latter is called the p-dimensional infinite cohomology group of $\mathfrak{S}$ and the elements of $Z_p(\mathfrak{S})$ are called infinite cocycles.

If Υ^p is a compact cycle and Γ_q an infinite cocycle, $q \leqq p$, the collection $\{\Upsilon_\lambda^p \cdot \Gamma_q{}^\lambda\}$ is easily seen to be a compact cycle, Υ^{n-p}, and we define this to be the intersection, $(\Upsilon^p \cdot \Gamma_q)$ of Υ^p and Γ_q.

The Kronecker Index, $KI(\Upsilon^p, \Gamma_p)$, induces a pairing of $h^p(\mathfrak{S})$ and $H_p(\mathfrak{S})$ to the coefficient field, and, as Lefschetz has shown, these two groups are orthogonal under this pairing. Hence [15, pp. 262, 77]:

THEOREM L_1. *If either one of the groups $h^p(\mathfrak{S})$, $H_p(\mathfrak{S})$ has a finite basis, there is an isomorphism θ_1 of $h^p(\mathfrak{S})$ onto $H_p(\mathfrak{S})$ such that if Υ^p is any basis element of $h^p(\mathfrak{S})$, then $KI(\Upsilon^p \cdot \theta_1(\Upsilon^p)) = 1$.*

Paralleling these groups are the groups of compact cocycles and infinite cycles of $\mathfrak{S}$, denoted by $h_p(\mathfrak{S})$ and $H^p(\mathfrak{S})$. These are defined in the same manner as the above groups, but with the family $\{L_\lambda\}$ replaced by the family $\{O_\lambda\}$ of all open subsets of $\mathfrak{S}$ with compact closures. For the details see [15, p. 262]. In this case a compact cocycle is a direct system and is defined up to cohomology by any one of its coordinates. Here also the intersection of Γ^p and Υ_q, $q \leqq p$, can be defined and is a compact cycle of dimension $p - q$.

[5] A set L carries a cycle Γ if all coordinates of Γ, as well as all chains used in establishing the homologies between different coordinates, are on L (in the sense of [B, p. 561]). L is called a carrier of Γ. A similar definition holds for cocycles, the carrier this time being an open set.

It is clear that if $\mathfrak{S}$ is compact, then $h^p(\mathfrak{S}) \simeq H^p(\mathfrak{S})$ and $h_p(\mathfrak{S}) \simeq H_p(\mathfrak{S})$.

THEOREM L_2. *If either one of the groups $H^p(\mathfrak{S})$, $h_p(\mathfrak{S})$ has a finite basis, there is an isomorphism θ_2 of $H^p(\mathfrak{S})$ onto $h_p(\mathfrak{S})$ such that if Γ^p is a basis element of $H^p(\mathfrak{S})$, then $KI(\Gamma^p \cdot \theta_2(\Gamma^p)) = 1$.*

Since these isomorphisms do not exist in the case of infinite bases, our results below hold only for groups with finite bases. It is therefore understood without further mention that all the homology groups below are assumed to have finite bases, and all Betti numbers and local Betti numbers are assumed to be finite.

In many cases the locally compact space $\mathfrak{S}$ appears as an open subset of another locally compact space $\mathfrak{T}$, and it is desirable to relate the groups of $\mathfrak{S}$ with those of $\mathfrak{T}$. Denote by $H^p(\mathfrak{T} \bmod (\mathfrak{T} - \mathfrak{S}))$ the group of p-cycles of $\mathfrak{T} \bmod (\mathfrak{T} - \mathfrak{S})$ reduced modulo those which are $\sim 0 \bmod (\mathfrak{T} - \mathfrak{S})$. Similarly, let $H_p(\mathfrak{T} \bmod (\mathfrak{T} - \mathfrak{S}))$ be the group of p-cocycles of $\mathfrak{T}$ which are in $\mathfrak{S}$ reduced modulo those which are ~ 0 in $\mathfrak{S}$.

THEOREM 1.1. *If the closure of $\mathfrak{S}$ in $\mathfrak{T}$ is compact, then $H_p(\mathfrak{T} \bmod (\mathfrak{T} - \mathfrak{S}))$ is isomorphic to $h_p(\mathfrak{S})$ and $H^p(\mathfrak{T} \bmod (\mathfrak{T} - \mathfrak{S}))$ is isomorphic to $H^p(\mathfrak{S})$.*

Proof. The second part of this theorem follows from Theorem L_2 and from Theorem (4.7) of Chapter VII of [15]. Hence we need only prove the first isomorphism.

Let then Γ_p be a cocycle of $\mathfrak{T}$ which lies in $\mathfrak{S}$ and let $\mathfrak{U}$ be a covering (all coverings are understood to be finite) of $\mathfrak{T}$ such that Γ_p has a coordinate $\Gamma_p{}^u$ on U. By a theorem due to A. D. Wallace [15, p. 263], there is a refinement $\mathfrak{V}$ of $\mathfrak{U}$ and a neighborhood O of $\mathfrak{T} - \mathfrak{S}$ such that $\pi^*{}_v{}^u\Gamma_p{}^u$ is on $\mathfrak{T} - O$, a compact subset of $\mathfrak{S}$. In the same way, if Υ_p is a compact cocycle of $\mathfrak{S}$ such that $\Upsilon_p \sim 0$ in $\mathfrak{S}$, then $\pi^*{}_v{}^u\Upsilon_p{}^u \sim O$ in a compact subset of $\mathfrak{S}$. This proves the theorem.

We recall that in definitions of local connectedness a 0-chain is to be considered a 0-cycle only if the sum of its coefficients is zero. In defining local co-Betti numbers, it is necessary to make an analogous convention regarding 0-cocycles. The most convenient method of doing this is to augment [6] all complexes involved, i. e. to add to each complex a new element of dimension -1 which has with each 0-simplex the incidence number 1. The homology and cohomology groups based on augmented complexes are denoted

[6] For a complete discussion of this notion, see [15, p. 130].

by $H_a{}^p$, $H_p{}^a$ etc. Clearly, for $p > 0$, H^p and $H_a{}^p$ are the same. However, $H_a{}^0$ has one less generator than H^0, and similarly for $H_0{}^a$ and H_0. The duality theorems above still hold in the augmented case.

In the following, in all definitions of local connectedness, local Betti numbers, and local co-Betti numbers, all the complexes involved are understood to be augmented. In all other cases all complexes are unaugmented unless the contrary is explicitly stated.

2. Local connectedness in locally compact spaces. For such spaces, Čech's definition of local connectedness [8] is most useful.

Definition 2.1. $\mathfrak{S}$ is q—lc at a point s if given a neighborhood P of s and a covering $\mathfrak{B}$ of $\mathfrak{S}$, there exists a neighborhood $Q = Q^q(P) \subset P$ and a covering $\mathfrak{A} = \mathfrak{A}^q(\mathfrak{B}, P) < \mathfrak{B}$ such that if X^q is a cycle on $A \cap Q$, then $X^q = FY^{q+1}$, where Y^{q+1} is a chain of $(A \cup B) \cap P$. $\mathfrak{S}$ is q—lc if it is q—lc at every point, and $\mathfrak{S}$ is lc^p if it is q—lc for all $q \leq p$.

Definition 4.2 of [B] is a definition of uniform local connectedness, and hereafter the notations q—ulc, ulc^p, etc., will be used for this property. If $\mathfrak{S}$ is compact, then definitions 4.2 of [B] and 2.1 are equivalent, but in general this is not the case. In particular, Theorem 5.5 of [B] is not a consequence of Definition 2.1. However, we can prove an analogous but weaker theorem which is sufficient for our purposes.

THEOREM 2.1. *Let L be a compact subset of $\mathfrak{S}$ and O a neighborhood of L. If $\mathfrak{S}$ is lc^p, then for each pair $\mathfrak{C}$ and $\mathfrak{B}$ of coverings of $\mathfrak{S}$, there exist coverings $\mathfrak{H} = \bar{\mathfrak{H}}^p(\mathfrak{C}; L, O)$ and $\mathfrak{A} = \bar{\mathfrak{A}}^p(\mathfrak{B}, \mathfrak{C}; L, O)$ such that if K is a complex of dimension $\leqq p + 1$ and τ' is a partial realization of K on $A \cap L$ of norm $< \mathfrak{H}$, then τ' can be extended to a realization τ of K on $(A \cup B) \cap O$ of norm $\mathfrak{C}$.*

Proof. For each $q \leq p$ and each compact subset L of S, it follows, by the usual argument, that for each covering $\mathfrak{C}$ of $\mathfrak{S}$ there is a refinement $\mathfrak{H}^q(\mathfrak{C}; L)$ and for each covering $\mathfrak{B}$ there is a refinement $\mathfrak{A}^q(\mathfrak{B}, \mathfrak{C}; L)$ such that if X^q is a cycle of $\mathfrak{A} \cap L$ of diameter $< \mathfrak{H}^q(\mathfrak{C}, L)$, then $X^q = FY^{q+1}$ where Y^{q+1} is a chain on $A \cup B$ of diameter $< \mathfrak{C}$.

Now suppose that the theorem has been proved for $q - 1$. Choose an open set O_1 such that $L \subset O_1 \subset \bar{O}_1 \subset O$ and such that $\bar{O}_1$ is compact. Given $\mathfrak{C}$, choose $\mathfrak{C}_1 < \mathfrak{C}$ such that $St(\bar{O}_1, \mathfrak{C}_1) \subset O$. Then choose $\mathfrak{C}_2 \mathbin{*<} \mathfrak{H}^q(\mathfrak{C}_1, \bar{O}_1)$ and set $\bar{\mathfrak{H}}^q(\mathfrak{C}; L, O) = \bar{\mathfrak{H}}^{q-1}(\mathfrak{C}_2; L, O_1)$. Given $\mathfrak{B}$, let $\mathfrak{B}_1 = \mathfrak{A}^q(\mathfrak{B}, \mathfrak{C}_1; \bar{O}_1)$ and set $\bar{\mathfrak{A}}^q(\mathfrak{B}, \mathfrak{C}; L, O) = \bar{\mathfrak{A}}^{q-1}(\mathfrak{B}_1, \mathfrak{C}_2; L, O_1)$.

That these choices of $\mathfrak{H}$ and $\mathfrak{A}$ are satisfactory follows by essentially the same argument as used in the proof of Theorem 1.1 of [B].

The proof of Theorem 5.5 of [B], using Theorem 2.1 in the appropriate places, now yields

THEOREM 2.2. *Let $\mathfrak{S}$ be lc^p, L a compact subset of $\mathfrak{S}$, and O a neighborhood of L. Then there is a covering $\mathfrak{U}_0$ of $\mathfrak{S}$ such that if Γ^q and $\bar{\Gamma}^q$, $q \leq p$, are cycles of $\mathfrak{S}$ with $\Gamma_0{}^q \sim \bar{\Gamma}_0{}^q$ on $U_0 \cap L$, then $\Gamma^q \sim \bar{\Gamma}^q$ in O.*

The following corollary, which is an immediate consequence of the above theorem, generalizes a theorem due to Wilder [21] from locally compact metric spaces to locally compact spaces.

COROLLARY 2.3. *Let $\mathfrak{S}$ be lc^p, L a compact subset of $\mathfrak{S}$, and O any neighborhood of L. Then there are at most a finite number of p-cycles in L which are independent with respect to homologies in O.*

3. Open generalized manifolds.

Definition 3.1. A locally compact space $\mathfrak{M}$ is an open generalized manifold of dimension n if

a) $\dim \mathfrak{M} = n$,

b) $\mathfrak{M}$ is lc^n,

c) For each point $s \in \mathfrak{M}$, $R_p(s) = 0$, $p < n$,

d) For each point $s \in \mathfrak{M}$, $R_n(s) = 1$.

$\mathfrak{M}$ is said to be orientable if there is an infinite n-cycle Γ^n on $\mathfrak{M}$ which is not homologous to any cycle on any proper closed subset of $\mathfrak{M}$.

Note that any open subset of an open generalized n-manifold, or of a compact generalized n-manifold, is itself an open generalized n-manifold.

4. Cochain realizations. Let K be a finite complex of $\dim \leq n$ and let U be the nerve of a covering $\mathfrak{U}$ of the open orientable generalized manifold, $\mathfrak{M}$. A function τ^* which assigns to each chain X^q of K a cochain $\tau^*(X^q) = X_{n-q}$ of U is called a cochain realization of K on U if

a) τ^* is a linear mapping,

b) $\tau^*(FX^q) = F\tau^*X^q$, where F on the left is the boundary operator and on the right is the coboundary operator.

c) $KI(X^0) = KI(\Gamma^n \cdot \tau^*X^0)$ for every 0-chain X^0 on K.

The last condition has meaning since by b) the image under τ^* of a 0-chain (necessarily a 0-cycle) of K is an n-cocycle of U and hence of $\mathfrak{M}$.

The definition of a partial cochain realization $\tau^{*\prime}$ is now parallel to definition 1. 4 of [b]. The norm of τ^* or $\tau^{*\prime}$ is defined in the obvious manner.

THEOREM 4. 1. *Let $\mathfrak{M}$ be an orientable open generalized n-manifold. Let L be a compact subset of $\mathfrak{M}$ and O a neighborhood of L with a compact closure. For each covering $\mathfrak{E}$ of $\mathfrak{M}$ there is a refinement $\mathfrak{H}_n(\mathfrak{E}; L, O)$ and for each covering $\mathfrak{B}$ of $\mathfrak{M}$ there is a refinement $\mathfrak{A}_n(\mathfrak{B}, \mathfrak{E}; L, O)$ such that if $\tau^{*\prime}$ is a partial cochain realization on $B \cap L$ of norm $< \mathfrak{H}_n(\mathfrak{E}; L, O)$ of an n-dimensional complex K, then there is a cochain realization $\tau^* X^q = \pi^*{}_a{}^b \tau^{*\prime} X^q$ whenever the latter is defined.*

The proof of this theorem is entirely parallel to that of Theorem 2. 1 using conditions c) and d) of definition 3. 1 in place of local connectedness.

5. The Poincaré duality theorem. For open generalized manifolds this theorem takes the following form:

THEOREM 5. 1. *If $\mathfrak{M}$ is an orientable open generalized n-manifold, then $H^p(\mathfrak{M}) \simeq h^{n-p}(\mathfrak{M})$.*

Proof. By Theorem L_2, $H^p(\mathfrak{M})$ is isomorphic to $h_p(\mathfrak{M})$. Let Υ_p be any compact cocycle of $\mathfrak{M}$. Then $(\Gamma^n \cdot \Upsilon_p) = \Upsilon^{n-p}$ is a compact cycle of $\mathfrak{M}$ which we denote by $\phi(\Upsilon_p)$. Clearly ϕ induces a homomorphism of $h_p(\mathfrak{M})$ into $h^{n-p}(\mathfrak{M})$. We show first that $\phi(h_p(\mathfrak{M}))$ covers $h^{n-p}(\mathfrak{M})$.

Let Υ^{n-p} be a compact cycle of $\mathfrak{M}$ and L a compact subset of $\mathfrak{M}$ which carries it. Let O_1, O_2 and O_3 be neighborhoods of L in $\mathfrak{M}$ with compact closures such that $O_1 \supset \bar{O}_2 \supset O_2 \supset \bar{O}_3$. Let $\mathfrak{E}$ be a covering of $\mathfrak{M}$ such that $St(\bar{O}_3, \mathfrak{E}) < O_2$ and choose $\mathfrak{E}_1 * < \mathfrak{H}^n(\mathfrak{E}, \bar{O}_3, O_2)$ such that $St(L, \mathfrak{E}_1) \subset O_3$. Let $\mathfrak{U}_0$ be the covering of Theorem 2. 2 corresponding to the sets O_1 and $\bar{O}_2$ and let $\mathfrak{U}_1$ be an n-dimensional refinement of $\mathfrak{U}_0$ which is $* < \mathfrak{H}_n(\mathfrak{E}_1; L, O_3)$ and $< \mathfrak{A}^n(\mathfrak{U}_0, \mathfrak{E}; \bar{O}_3, O_2)$. In each open set of $\mathfrak{U}_1$ there is a compact n-cocycle of $\mathfrak{M}$ whose intersection with Γ^n has Kronecker index equal to 1. We may assume that these cocycles each have a coordinate on a U_2 where $\mathfrak{U}_2 < \mathfrak{U}_1$. Now we define $\tau^{*\prime}$ by letting $\tau^{*\prime}$ assign to each vertex of U_1 the corresponding n-cocycle of U_2. $\tau^{*\prime}$ satisfies the conditions of Theorem 4. 1, so there is a cochain realization τ^* of $U_1 \cap L$ of norm $< \mathfrak{E}_1$ on $U_3 \cap O_3$ where $\mathfrak{U}_3 = \mathfrak{A}_n(\mathfrak{U}_2, \mathfrak{E}_1; L, O_3)$. In particular, if $x_1{}^{n-p}$ is the coordinate on U_1 of Υ^{n-p}, then $\tau^* x_1{}^{n-p} = x_p{}^3$ is a cocycle of $U_3 \cap O_3$, and $x_p{}^3$ is in O_3. Hence $x_p{}^3 = \Upsilon_p$ is a compact cocycle of $\mathfrak{M}$.

We wish to show that $(\Gamma^n \cdot \Upsilon_p) \sim \Upsilon^{n-p}$ in O_1. To this end, let K be the product of $U_1 \cap \bar{O}_3$ with the unit interval, simplicially subdivided so that all the vertices of K are on the base and the top of K. We identify $U_1 \cap \bar{O}_3$ with the base of K. Note that if x^q is a cycle of $U_1 \cap \bar{O}_1$ and if $\bar{x}^q$ is the corresponding cycle on the top of K, then there is a chain y^{q+1} in K such that $Fy^{q+1} = x^q - \bar{x}^q$.

We define now a partial realization τ' of K as follows: on the base of K, τ' is the identity mapping and for any chain $\bar{x}^q$ on the top of K, $\tau'\bar{x}^q = (-1)^{nq}\pi_3{}^1(X_3{}^n \cdot \tau^* x^q)$, where $X_3{}^n$ is a coordinate of Γ^n on U_3. To show that τ' is a partial realization it is only necessary to show that $\tau' F\bar{x}^q = F\tau'\bar{x}^q$ (see definition 1.2 of [B]).[7] But we have, for any $\bar{x}^q$ on the top of K,

$$
\begin{aligned}
F\tau'\bar{x}^q = F[(-1)^{nq}\pi_3{}^1(X_3{}^n \cdot \tau^* x^q)] &= (-1)^{nq}\pi_3{}^1 F(X_3{}^n \cdot \tau^* x^q) \\
&= (-1)^{nq}\pi_3{}^1[(FX_3{}^n \cdot \tau^* x^q) + (-1)^n(X_3{}^n \cdot F\tau^* x^q)] \\
&= (-1)^{n(q+1)}\pi_3{}^1(X^n \cdot \tau^* F x^q).
\end{aligned}
$$

Also, $\tau' F x^q = (-1)^{n(q-1)}\pi_3{}^1(X_3{}^n \cdot \tau^* F x^q)$. Since $n(q+1)$ and $n(q-1)$ are odd or even together, τ' is a chain mapping and hence a partial realization. Since $\mathfrak{E}_1 *< \mathfrak{H}^n(\mathfrak{E}, \bar{O}_3, O_2)$, norm $\tau' < \bar{\mathfrak{H}}^n(\mathfrak{E}; \bar{O}_3, O_2)$. Therefore, by the choice of $\mathfrak{U}_1 < \bar{\mathfrak{A}}^n(\mathfrak{U}_0, \mathfrak{E}; \bar{O}_3, O_2)$, τ' can be extended to a realization τ of K on $(U_1 \cup U_0) \cap O_2$.

Now let y^{n-p-1} be a chain on K such that $Fy^{n-p-1} = x_1{}^{n-p} - \bar{x}_1{}^{n-p}$, where $x_1{}^{n-p}$ is the coordinate of Υ^{n-p} on U_1. Then $F\tau y^{n-p-1} = x_1{}^{n-p} - \pi_3{}^1(X_3{}^n \cdot \tau^* x_1{}^{n-p})$ and τy^{n-p-1} is on $(U_1 \cup U_0) \cap O_2$. Let π be the natural mapping of $(U_1 \cup U_0)$ into U_0. Then $F\pi\tau y^{n-p-1} = \pi_1{}^0 x_1{}^{n-p} - \pi_1{}^0\pi_3{}^1(X_3{}^n \cdot \tau^* x_1{}^{n-p})$. Thus the coordinates of Υ^{n-p} and $(\Gamma^n \cdot \Upsilon_p)$ on U_0 are homologous on $\bar{O}_2$. But by Theorem 2.2, this is sufficient to show that $(\Gamma^n \cdot \Upsilon_p) \sim \Upsilon^{n-p}$ in $\bar{O}_1$. Hence ϕ is a homomorphism of $H^p(\mathfrak{M})$ onto $h^{n-p}(\mathfrak{M})$.

To complete the proof, it is sufficient to show that there is a homomorphism ψ of $h^{n-p}(\mathfrak{M})$ onto $H^p(\mathfrak{M})$. The existence of such a homomorphism is demonstrated by the same methods as used above but starting with $h^{n-p}(\mathfrak{M})$ and Theorem L_1. The details are left to the reader.

Remark. The above proof shows that the two groups involved have the same number of generators. Since the coefficient group is a field, this means that the two groups are isomorphic. However, since each one of the groups

[7] In a similar construction in the proof of Theorem 7.3 of [B], the factor $(-1)^{nq}$ was omitted. It should be inserted there.

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has a finite basis, it is easy to see that the homomorphism ϕ constructed above is itself an isomorphism.

In the above proof we have assigned to each compact cycle Υ^{n-p} a compact cocycle Υ_p, which we denote by $\tau^*\Upsilon^{n-p}$, in such a way that $(\Gamma^n \cdot \tau^*\Upsilon^{n-p}) \sim \Upsilon^{n-p}$. If $\Upsilon^{n-p} \sim 0$ on a set L_1, then there is a chain y^{n-p-1} on $U_1 \cap L_1$ (same notation as above) such that $Fy^{n-p-1} = x_1^{n-p}$. Then $\tau^*y^{n-p-1} = y^3{}_{p+1}$ is a cochain on U_3 such that $Fy^3{}_{p+1} = \tau^*Fy_1{}^{n-p-1} = \tau^*x_1{}^{n-p} = x_p{}^3$. Hence $\tau^*\Upsilon^{n-p} \sim 0$, and it is clear that this cohomology can be made to take place in any assigned neighborhood of L_1. We gather these results in the following lemma for future use.

LEMMA 5.2. *Let $\mathfrak{M}$ be an orientable open generalized n-manifold and let Υ^{n-p} be a compact cycle on a compact set L and ~ 0 on a compact set L_1. Then for any choice of neighborhoods O and O_1 of L, L_1, there is a compact cocycle $\Upsilon_p = \tau^*\Upsilon^{n-p}$ in O such that $(\Gamma^n \cdot \Upsilon_p) \sim \Upsilon^{n-p}$ in O and $\Upsilon_p \sim 0$ in O_1.*

6. The Alexander and other duality theorems. In Section **6** of [B] we reproduced Čech's definition of local Betti numbers and also gave a definition of local co-Betti numbers. Now suppose that S is a closed subset of a space $\mathfrak{X}$ and that s is a point of S. Then there are local Betti and co-Betti numbers of the complement of S at the point s. These are defined in the same way as the local Betti and co-Betti numbers of S at the point s but with the neighborhoods P and Q of s replaced by $P - S$ and $Q - S$. These numbers are denoted by $R^q(\mathfrak{X} - S, s)$ and $R_q(\mathfrak{X} - S, s)$. We shall hereafter use $R^q(S, s)$ and $R_q(S, s)$ to denote the local Betti and co-Betti numbers of S at s.

THEOREM 6.1. *For each point s of S, $R^q(\mathfrak{X} - S, s) = R_q(\mathfrak{X} - S, s)$.*

The proof is exactly the same as that of Theorem 6.2 of [B].

THEOREM 6.2. *If $R_q(\mathfrak{X}, s) = R_{q+1}(\mathfrak{X}, s) = 0$, then $R_{q+1}(\mathfrak{X} - S, s) = R_q(S, s)$.*

Proof. We first show that $R_q(S, s) \leq R_{q+1}(\mathfrak{X} - S, s)$. Let O be any neighborhood of s in $\mathfrak{X}$. Since $R_q(\mathfrak{X}, s) = 0$, O contains a neighborhood P of s such that $R_q(O, P, s) = 0$. Let Q be any neighborhood of s which is in P.

Now let $\Gamma_q{}^1, \cdots, \Gamma_q{}^k$ be cocycles of S in $Q \cap S$ which are independent in $O \cap S$. Let $\mathfrak{U}_\zeta$ be a covering of $\mathfrak{X}$ such that each $\Gamma_q{}^i$ has a coordinate $X_q{}^{i\zeta}$ on U_ζ. Then $FX_q{}^{i\zeta}$ is a $(q+1)$-dimensional cocycle in $Q - S$. Let us suppose that these cocycles are not independent in $P - S$. Then there is a chain $Y_q{}^\rho$ on U_ρ, for some $\mathfrak{U}_\rho < \mathfrak{U}_\zeta$, such that $Y_q{}^\rho$ is in $P - S$ and $FY_q{}^\rho = \pi^*{}_\rho{}^\zeta \Sigma f_i(FX_q{}^{i\zeta})$. Then $(Y_q{}^\rho - \pi^*{}_\rho{}^\zeta \Sigma f_i X_q{}^{i\zeta})$ is a cocycle of $\mathfrak{X}$ in P.

By the choice of P, this cocycle must bound in O. Hence there is a chain $Z^{\mu}{}_{q-1}$ on U_{μ}, for some $\mathfrak{U}_{\mu} < \mathfrak{U}_{\rho}$, such that $Z^{\mu}{}_{q-1}$ is in O and $FZ^{\mu}{}_{q-1} = \pi^*{}_{\mu}{}^{\rho} Y_q{}^{\rho} - \pi^*{}_{\mu}{}^{\rho}\pi^*{}_{\rho}{}^{\zeta}\Sigma f_i X_q{}^{i\zeta}$. But since $Y_q{}^{\rho}$ is in $O - S$, we have $\Sigma f_i \Gamma_q{}^i \sim 0$ in $O \cap S$, which is impossible. Therefore the cocycles $FX_q{}^{i\zeta}$ are independent in $P - S$, and hence we have $R_q(O \cap S, Q \cap S, s) \leq R_{q+1}(P - S, Q - S, s)$. From this we have immediately that $R_q(S, s) \leq R_{q+1}(\mathfrak{X} - S, s)$.

To prove the reverse inequality, let O be an arbitrary neighborhood of s. Since $R_{q+1}(\mathfrak{X}, S) = 0$, there are neighborhoods $P \supset Q$ of s contained in O such that $R_{q+1}(P, Q, s) = 0$.

Now let $\Gamma^1{}_{q+1}, \cdots, \Gamma^k{}_{q+1}$ be cocycles in $Q - S$ which are independent in $O - S$. By the choice of Q with relation to P, each of these cocycles bounds in P. Hence there is a covering $\mathfrak{U}_{\zeta}$ of $\mathfrak{X}$ such that each $\Gamma^i{}_{q+1}$ has a coordinate $X^{i\zeta}{}_{q+1}$ on U_{ζ}, and there are chains $Y_q{}^{i\zeta}$ on U_{ζ} such that $Y_q{}^{i\zeta}$ is in P and $FY_q{}^{i\zeta} = X^{i\zeta}{}_{q+1}$. Let $Y_q{}^{i\zeta} = A_q{}^{i\zeta} + B_q{}^{i\zeta}$, where $A_q{}^{i\zeta}$ is the part of $Y_q{}^{i\zeta}$ in $\mathfrak{X} - S$. Then each $B_q{}^{i\zeta}$ is a q-dimensional cocycle of S in $P \cap S$. Let us suppose that these cocycles are not independent in $O \cap S$. Then on some U_{ρ}, $\mathfrak{U}_{\rho} < \mathfrak{U}_{\zeta}$, there is a chain $Z^{\rho}{}_{q-1}$ in O such that $FZ^{\rho}{}_{q-1} = \pi^*{}_{\rho}{}^{\zeta}\Sigma f_i B_q{}^{i\zeta} + C_q{}^{\rho}$, where $C_q{}^{\rho}$ is in $O - S$. Then we have $0 = FFZ^{\rho}{}_{q-1} = FC_q{}^{\rho} + \pi^*{}_{\rho}{}^{\zeta}\Sigma f_i(FB_q{}^{i\zeta}) = FC_q{}^{\rho} + \pi^*{}_{\rho}{}^{\zeta}\Sigma f_i(F(Y_q{}^{i\zeta} - A_q{}^{i\zeta})$. Then $\pi^*{}_{\rho}{}^{\zeta}\Sigma f_i FY_q{}^{i\zeta} = \pi^*{}_{\rho}{}^{\zeta}\Sigma f_i X^{i\zeta}{}_{q+1} = F(C_q{}^{\rho} + \pi^*{}_{\rho}{}^{\zeta}\Sigma f_i A_q{}^{i\zeta})$. But $C_q{}^{\rho}$ is in $O - S$ and so is each $A_q{}^{i\zeta}$. Hence $\Sigma f_i \Gamma_{q+1} \sim 0$ in $O - S$, which is impossible. Therefore the cocycles $B_q{}^{i\zeta}$ are independent in $O \cap S$, so we have $R_{q+1}(O - S, Q - S, s) \leq R_q(O \cap S, P \cap S, s)$. From this we have immediately that $R_{q+1}(\mathfrak{X} - S, s) \leq R_q(S, s)$ which completes the proof.

THEOREM 6.3.[8] *If $h_q{}^a(\mathfrak{X}) = h^a{}_{q+1}(\mathfrak{X}) = 0$, then $h_q{}^a(S) = h^a{}_{q+1}(\mathfrak{X} - S)$.*

Proof. Let $\Upsilon_q{}^1, \cdots, \Upsilon_q{}^k$ be independent compact cocycles of S, and let $\Gamma_q{}^i$ be a Čech cocycle which is a component of $\Upsilon_q{}^i$. Let G be an open set whose closure in $\mathfrak{X}$ is compact and which carries all the $\Gamma_q{}^i$. Repeating the construction in the first part of the proof of the above theorem, we obtain the cocycles $FX_q{}^{i\zeta}$. By a suitable choice of $\mathfrak{U}_{\rho}$, these cocycles all lie in an open set $G' \subset \mathfrak{X} - S$ whose closure in $\mathfrak{X}$ is compact. Hence we may use Theorem 1.1 to obtain compact cocycles $x^i{}_{q+1} \sim FX_q{}^{i\zeta}$ in G'. Also, by the argument above taken in conjunction with Theorem 1.1, these compact cocycles are independent in $\mathfrak{X} - S$. Thus rank $(h_q{}^a(S)) \leq$ rank $(h^a{}_{q+1}(\mathfrak{X} - S))$. The reverse inequality is proved in an analogous fashion.

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[8] See pp. 227-229 of [15]. Also, see P. Alexandroff, "General combinatorial topology," *Transactions of the American Mathematical Society*, vol. 49 (1941), pp. 41-105, where this theorem is called Kolmogoroff's duality law.

As in the case of Theorem 5.1, the particular construction used above leads to an isomorphism since each one of the two groups has a finite basis.

We are now ready to prove our main theorem.

THEOREM 6.4. [Alexander's Duality Theorem]. *Let $\mathfrak{M}$ be an orientable open connected n-dimensional generalized manifold which has the homology groups of the n-sphere. If L is any closed subset of $\mathfrak{M}$, all of whose Betti numbers are finite, then* $H_a{}^q(L) \cong h_a{}^{n-q-1}(\mathfrak{M} - L)$ *for* $q \leq n - 1$.

Proof. Since $\mathfrak{M}$ is connected and n-sphere-like, $H_a{}^q(\mathfrak{M}) = 0$ for $q \leq n - 1$. Hence $h_q{}^a(\mathfrak{M}) = 0$ for $q \leq n - 1$. Therefore, by Theorem 6.3, $H_a{}^q(L) \simeq h_q{}^a(L) \simeq h^a{}_{q+1}(\mathfrak{M} - L)$ for $q < n - 1$. Since $q + 1 > 0$, $h^a{}_{q+1}(\mathfrak{M} - L) = h_{q+1}(\mathfrak{M} - L)$, and by Theorem 5.1, $h_{q+1}(\mathfrak{M} - L) \simeq h^{n-q-1}(\mathfrak{M} - L)$. Since $q < n - 1$, $n - q - 1 > 0$, so $h^{n-q-1}(\mathfrak{M} - L) \simeq h_a{}^{n-q-1}(\mathfrak{M} - L)$, which completes the proof for $q < n - 1$.

For $q = n - 1$, let $\Upsilon^1{}_{n-1}, \cdots, \Upsilon^k{}_{n-1}$ be compact cocycles of L which are independent on L. As in the first part of the proof of Theorem 6.3, there are compact cocycles $\Upsilon_n{}^1, \cdots, \Upsilon_n{}^k$ in $\mathfrak{M} - L$ which are independent in $\mathfrak{M} - L$. By Theorem 5.1, to each $\Upsilon_n{}^i$ there corresponds a compact cycle $\Upsilon_i{}^0$ of $\mathfrak{M} - L$ in such a way that the $\Upsilon_i{}^0$ are independent in $\mathfrak{M} - L$ and furthermore $KI(\Upsilon_i{}^0) = KI(\Gamma^n \cdot \Upsilon_n{}^i)$. But since each $\Upsilon_n{}^i$ is ~ 0 in $\mathfrak{M}$, $KI(\Gamma^n \cdot \Upsilon_n{}^i) = 0$. Hence the $\Upsilon_i{}^0$ determine independent elements of $h_a{}^0(\mathfrak{M} - L)$ and so rank $(H_a{}^{n-1}(L)) \leq$ rank $(h_a{}^0(\mathfrak{M} - L))$.

Conversely, let $\Upsilon_1{}^0, \cdots, \Upsilon_k{}^0$ be independent compact cycles of $\mathfrak{M} - L$ with $KI(\Upsilon_i{}^0) = 0$ for each i. By Theorem 5.1, there are compact cocycles $\Upsilon_n{}^i$ of $\mathfrak{M} - L$, such that $(\Gamma^n \cdot \Upsilon_n{}^i) \sim \Upsilon_i{}^0$. Hence $KI(\Gamma^n \cdot \Upsilon_n{}^i) = 0$. It follows from Lefschetz's Theorem ([B] p. 568) that each $\Upsilon_n{}^i \sim 0$ in $\mathfrak{M}$. Now the second part of the proof of Theorem 6.2 together with Theorem 1.1 applies to show that there are at least k compact $(n - 1)$-cocycles of L which are independent on L. Hence rank $(h_a{}^0(\mathfrak{M} - L)) \leq$ rank $(H_a{}^{n-1}(L))$, and this is sufficient to show that $H_a{}^{n-1}(L) \simeq h_a{}^0(\mathfrak{M} - L)$, which completes the proof.

Čech [7] has shown that the above theorem can be localized in the following manner: Let $\mathfrak{T}$ be a locally compact space and S a closed subset of $\mathfrak{T}$. Let $P \supset Q$ be neighborhoods of a point s of S. Denote by $r^q(P - S, Q - S)$ the maximum number of compact q-cycles in $Q - S$ which are independent with respect to homologies in $P - S$. Then, using these numbers, define $r^q(\mathfrak{T} - S, s)$ in the obvious way.

THEOREM 6.5. *Let $\mathfrak{M}$ be an open generalized n-manifold and let L be a closed subset of $\mathfrak{M}$. Then for each point s of L,* $R^q(L, s) = r^{n-q-1}(\mathfrak{M} - L, s)$ *for all* $q \leq n - 1$.

Proof. Let $P \supset Q$ be neighborhoods of s such that $R_q(P,Q) = R_{q+1}(P,Q) = 0$ and $R_n(P,Q) = 1$. Then, since L is closed in $\mathfrak{M}$, $P - L$ and $Q - L$ are open orientable generalized n-manifolds. Now, by trivial modifications in the proofs of Theorems 6.2 and 5.1, we have $R^q(P \cap L, Q \cap L) = R_q(P \cap L, Q \cap L) = R_{q+1}(P - L, Q - L) = r^{n-q-1}(P - L, Q - L)$. This proves the theorem.

As an immediate consequence we have the following result on uniform local connectedness.[9]

COROLLARY 6.6. *Let $\mathfrak{M}$ be a compact generalized n-manifold and let L be a closed subset of $\mathfrak{M}$. If $R^q(L,s) = 0$ for every point s of L, then $\mathfrak{M} - L$ is $(n - q - 1)$ — ulc.*

7. Geometric linking. Let S be a closed subset of a compact space $\mathfrak{T}$. A cycle Γ^p of S and a compact cycle Υ^q of $\mathfrak{T} - S$ are said to be *linked* if neither one is homologous to zero (as a compact cycle) in the complement of any carrier[5] of the other. If (Γ^p) is the element of $H^p(S)$ containing Γ^p and (Υ^q) the element of $h^q(\mathfrak{T} - S)$ containing Υ^q, we say that (Γ^p) and (Υ^q) are linked if any element of one class is linked with any element of the other.

THEOREM 7.1. *Let $\mathfrak{M}$ be a compact orientable generalized n-manifold which is n-sphere-like, and let L be a closed subset of $\mathfrak{M}$ all of whose Betti numbers are finite. Then in the isomorphism of $H^p(L)$ and $h^{n-p-1}(\mathfrak{M} - L)$ established in Theorem 6.4, corresponding elements are linked.*

Proof.[10] If we trace the isomorphisms involved in the proof of Theorem 6.4, we see that to each element (Γ^p) of $H^p(L)$ there is assigned an element (Υ^{n-p-1}) of $h^{n-p-1}(\mathfrak{M} - L)$ in the following way: an arbitrary cycle Γ^p is chosen in (Γ^p) and a cocycle Γ_p of L is given by Theorem L_1 such that $KI(\Gamma^p \cdot \Gamma_p) \neq 0$. Then $\{F\Gamma_p{}^{\zeta}\} = \{A^{\zeta}{}_{p+1}\}$ is a cocycle of $\mathfrak{M} - L$. By Theorem 1.1, $\{A^{\zeta}{}_{p+1}\}$ is cohomologous to a compact cocycle Υ_{p+1} of $\mathfrak{M} - L$. Then $(\Gamma^n \cdot \Upsilon_{p+1}) = \Upsilon^{n-p-1}$ is a compact cycle of $\mathfrak{M}$ — L and the class (Υ^{n-p-1}) is the one assigned to (Γ^p).

Now let Γ^p be any cycle of L and let Υ^{n-p-1} be the compact cycle of $\mathfrak{M} - L$ which is obtained from Γ^p by the above process, and let $\bar{\Upsilon}^{n-p-1}$ be any element of (Υ^{n-p-1}). If we can show that Γ^p and Υ^{n-p-1} are linked, the theorem will be proved.

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[9] Cf. Theorem 1 of [20].

[10] Cf. L. Pontrjagin, "Zum Alexanderschen Dualitätssatz," *Göttingen Nachrichten* (1927), pp. 315-329. I am indebted to Professor R. L. Wilder both for the suggestion that this theorem might hold and for an outline of essentially the proof used here.

First let us suppose that there is a carrier $[\bar{\Upsilon}^{n-p-1}]$ of $\bar{\Upsilon}^{n-p-1}$ such that $\Gamma^p \sim 0$ on a compact set T which does not meet $[\bar{\Upsilon}^{n-p-1}]$. Then there is a neighborhood O of $[\bar{\Upsilon}^{n-p-1}]$ whose closure is compact, is contained in $\mathfrak{M} - L$, and does not meet T. By Lemma 5.2, there is a compact cocycle $\bar{\Upsilon}_{p+1} = \tau^*\bar{\Upsilon}^{n-p-1}$ in O such that $(\Gamma^n \cdot \bar{\Upsilon}_{p+1}) \sim \bar{\Upsilon}^{n-p-1}$ in O. Since $H^p(L)$ has a finite basis, the homomorphism $\Upsilon_{p+1} \to (\Gamma^n \cdot \Upsilon_{p+1})$ is an isomorphism. Therefore, since $\Upsilon^{n-p-1} \sim \bar{\Upsilon}^{n-p-1}$ in $\mathfrak{M} - L$, $\Upsilon_{p+1} \sim \bar{\Upsilon}_{p+1}$ in $\mathfrak{M} - L$, where Υ_{p+1} is the compact cocycle above such that $\Upsilon^{n-p-1} = (\Gamma^n \cdot \Upsilon_{p+1})$.

Let $\mathfrak{U}_\zeta$ be a covering of M such that Γ_p has a coordinate on U_ζ. Let $\mathfrak{U}_\rho$ be a refinement of $\mathfrak{U}_\zeta$ such that $\pi^*{}_\rho{}^\zeta F\Gamma_p{}^\zeta = \pi^*{}_\rho{}^\zeta A^\zeta{}_{p+1} \sim \Upsilon^\rho{}_{p+1}$ on U_ζ and such that $\Upsilon^\rho{}_{p+1} \sim \bar{\Upsilon}^\rho{}_{p+1}$ on U_ρ. Let $B_p{}^\rho$ in $\mathfrak{M} - L$ be such that $FB_p{}^\rho = \pi^*{}_\rho{}^\zeta A^\zeta{}_{p+1} - \Upsilon^\rho{}_{p+1}$, and let $C_p{}^\rho$ in $\mathfrak{M} - L$ be such that $FC_p{}^\rho = \bar{\Upsilon}^\rho{}_{p+1} - \Upsilon^\rho{}_{p+1}$. Then $F(\pi^*{}_\rho{}^\zeta\Gamma_p{}^\zeta - B_p{}^\rho + C_p{}^\rho) = \bar{\Upsilon}^\rho{}_{p+1}$. Since $[\bar{\Upsilon}_{p+1}]$ does not meet T, the part of $(\pi^*{}_\rho{}^\zeta\Gamma_p{}^\zeta - B_p{}^\rho + C_p{}^\rho)$ on $L \cup T$ is a cocycle Γ_p of $L \cup T$. Also, $KI(\Gamma^p \cdot \bar{\Gamma}_p) = KI(\Gamma_\rho{}^p \cdot \Gamma_p{}^\rho) = KI(\Gamma_\rho{}^p \cdot \pi^*{}_\rho{}^\zeta\Gamma_p{}^\zeta) = KI(\Gamma^p \cdot \Gamma_p)$, since $B_p{}^\rho$ and $C_p{}^\rho$ do not meet L. But $KI(\Gamma^p \cdot \Gamma_p) \neq 0$ while $KI(\Gamma^p \cdot \Gamma_p) = 0$, since $\Gamma^p \sim 0$ on $L \cup T$ and consequently has Kronecker index zero with every cocycle of $L \cup T$. This contradiction shows that Γ^p cannot bound in the complement of any carrier of $\bar{\Upsilon}^{n-p-1}$.

On the other hand, suppose that there is a closed set T such that $\bar{\Upsilon}^{n-p-1}$ bounds on T while T does not meet some carrier $[\Gamma^p]$ of Γ^p. Let O be an open set containing T with the closure of O not meeting $[\Gamma^p]$. By Lemma 5.2, there is a cocycle $\bar{\Upsilon}_{p+1}$ in $O \cap (\mathfrak{M} - L)$ such that $\bar{\Upsilon}_{p+1} \sim \Upsilon_{p+1}$ in $\mathfrak{M} - L$ as above. Also, since $\bar{\Upsilon}^{n-p-1} \sim 0$ in T, $\bar{\Upsilon}_{p+1} \sim 0$ in O. Now let $\mathfrak{U}_\zeta$, $\mathfrak{U}_\rho$, $B_p{}^\rho$, and $C_p{}^\rho$ be as above. We may also assume that on U_ρ there is a chain $D_p{}^\rho$ in O such that $FD_p{}^\rho = \bar{\Upsilon}^\rho{}_{p+1}$. Let the part of $D_p{}^\rho$ on L be $\bar{D}_p{}^\rho$, so that $\bar{D}_p{}^\rho$ is a coordinate of a cocycle $\bar{\Gamma}_p$ of L. Now $(\pi^*{}_\rho{}^\zeta\Gamma_p{}^\zeta - B_p{}^\rho + C_p{}^\rho) - D_p{}^\rho$ is a cocycle of U_ρ and hence of $\mathfrak{M}$. Since $\mathfrak{M}$ is n-sphere-like, this cocycle must be cohomologous to zero in $\mathfrak{M}$. Therefore there is a $\mathfrak{U}_\mu < \mathfrak{U}_\rho$ such that $\pi^*{}_\mu{}^\rho(\pi^*{}_\rho{}^\zeta\Gamma_p{}^\zeta - B_p{}^\rho + C_p{}^\rho) - \pi^*{}_\mu{}^\rho D_p{}^\rho = FE^\mu{}_{p-1}$. The part of $E^\mu{}_{p-1}$ on L is a chain $\bar{E}^\mu{}_{p-1}$ such that the part of $F\bar{E}^\mu{}_{p-1}$ which is on L is just $\pi^*{}_\mu{}^\rho\pi^*{}_\rho{}^\zeta\Gamma_p{}^\zeta - \pi^*{}_\mu{}^\rho\bar{D}_p{}^\rho$. Hence $\Gamma_p \sim \bar{\Gamma}_p$ on L, and consequently $KI(\Gamma^p \cdot \Gamma_p) = KI(\Gamma^p \cdot \bar{\Gamma}_p) \neq 0$. But $\bar{\Gamma}_p$ is in O, which does not meet $[\Gamma^p]$, so that the intersection of Γ^p and $\bar{\Gamma}_p$ must be vacuous. This contradiction completes the proof of the theorem.

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THE CONTACT OF A CUBIC SURFACE WITH A RULED SURFACE.*

By J. Ernest Wilkins, Jr.

1. Introduction. If a surface S in three-dimensional projective space is nonruled, it is known (see Lane [1]) that there exists a four-parameter family of cubic surfaces having contact of order four at a general point of S. The only nonruled surfaces, at each point of which there exists a single cubic surface with contact of order five, are the cubic surfaces, and at each point of a cubic surface there is a one-parameter family of cubic surfaces having contact of order five. It is the purpose of this paper to investigate these questions for ruled surfaces. We find it convenient to separate the discussion into two parts, according as S is developable or not. In the first case we shall prove the following

Theorem I. *At a general point of a developable surface, not a plane or a cone, there is a three-parameter family of cubic surfaces with contact of order five. There is a unique cubic surface with contact of order six and no cubic surface with contact of order seven.*

In case S is nondevelopable we shall prove the following

Theorem II. *At a general point of a nondevelopable ruled surface S, there is a one-parameter family of cubic surfaces with contact of order five. The only such surfaces for which at each point there exists a cubic surface with contact of order six are the cubic ruled surfaces and for these surfaces S the only cubic surface with sixth-order contact is the surface S itself.*

In order to prove these theorems we shall develop power series expansions for the surface S. These expansions are obtained from a system of partial differential equations which S must satisfy. It is hoped that other applications of these series will be made on another occasion.

I. Developable Surfaces.

2. The differential equations. It is known (see Lane [2, p. 98]) that

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the parametric vector equation of a developable surface not a plane or a cone can be taken in the form

$$x(u,v)=y'(v)+uy(v), \tag{2.1}$$

where $y(v)$ is the edge of regression of the developable and is a skew curve. Moreover, we may assume that the proportionality factor and the variable v have been chosen so that $y(v)$ satisfies the Laguerre-Forsythe canonical differential equation (see Lane [2, p. 84])

$$y''''=cy'-dy, \tag{2.2}$$

the quantities c and d being functions of v. Then it is easy to see that $x(u,v)$ is a solution of the completely integrable system of partial differential equations

$$\begin{aligned} &x_{uu}=0, \quad x_{uv}=x-ux_u,\\ &x_{vvv}=(u^3+c)x-(u^4+cu+d)x_u-u^2x_v+ux_{vv}. \end{aligned} \tag{2.3}$$

Conversely, if $x(u,v)$ is a solution of these three equations, the first two imply that x has the form (2.1) and then the third implies that y satisfies (2.2).

3. The power series expansions. Since S is not a plane, the points x, x_u, x_v, x_{vv} may be chosen as the vertices of a local tetrahedron of reference at the point x with the unit point so selected that a point $X=x_1x+x_2x_u+x_3x_v+x_4x_{vv}$ will have local homogeneous coördinates proportional to (x_1,x_2,x_3,x_4). If $X=x(u+\Delta u, v+\Delta v)$ is a point on S near x, we can employ a Taylor series and the differential equations (2.3) in a familiar fashion to express each of the local coördinates of X as a power series in Δu and Δv. These series, to as many terms as will be needed in the subsequent discussion, are as follows:

$$\begin{aligned} x_1&=1+\Delta u\Delta v-\tfrac{1}{2}u\Delta u\Delta v^2+\tfrac{1}{6}(u^3+c)\Delta v^3+\tfrac{1}{6}u^2\Delta u\Delta v^3\\ &\quad+\tfrac{1}{24}(c'-d)\Delta v^4+\cdots,\\ x_2&=\Delta u-u\Delta u\Delta v+\tfrac{1}{2}u^2\Delta u\Delta v^2-\tfrac{1}{6}(u^4+cu+d)\Delta v^3+\cdots,\\ x_3&=\Delta v+\tfrac{1}{2}\Delta u\Delta v^2-\tfrac{1}{6}u^2\Delta v^3-\tfrac{1}{6}u\Delta u\Delta v^3+\tfrac{1}{24}c\Delta v^4\\ &\quad+\tfrac{1}{120}(2c'-d)\Delta v^5+\cdots,\\ x_4&=\tfrac{1}{2}\Delta v^2+\tfrac{1}{6}u\Delta v^3+\tfrac{1}{6}\Delta u\Delta v^3+\tfrac{1}{120}c\Delta v^5\\ &\quad+\tfrac{1}{720}(3c'-d+cu)\Delta v^6+\cdots. \end{aligned} \tag{3.1}$$

If local nonhomogeneous coördinates are introduced by placing

$$x = x_2/x_1, \qquad y = x_3/x_1, \qquad z = x_4/x_1,$$

we find that they may be expressed in power series as follows:

$$
\begin{aligned}
&x = \Delta u - u\Delta u\Delta v - \Delta u^2\Delta v + \tfrac{1}{2}u^2\Delta u\Delta v^2 - \tfrac{1}{6}(u^4 + cu + d)\Delta v^3 + \cdots,\\
(3.2)\quad &y = \Delta v - \tfrac{1}{2}\Delta u\Delta v^2 - \tfrac{1}{6}u^2\Delta v^3 + \tfrac{1}{3}u\Delta u\Delta v^3 - \tfrac{1}{24}(4u^3 + 3c)\Delta v^4\\
&\quad + \tfrac{1}{2}\Delta u^2\Delta v^3 + \tfrac{1}{120}(4d - 3c')\Delta v^5 + \cdots,\\
&z = \tfrac{1}{2}\Delta v^2 + \tfrac{1}{6}u\Delta v^3 - \tfrac{1}{3}\Delta u\Delta v^3 + \tfrac{1}{12}u\Delta u\Delta v^4 - \tfrac{1}{120}(10u^3 + 9c)\Delta v^5\\
&\quad + \tfrac{1}{3}\Delta u^2\Delta v^4 - \tfrac{1}{720}(12c' - 14d + 19cu + 20u^4)\Delta v^6 + \cdots.
\end{aligned}
$$

It is now possible to calculate an expansion for z as a power series in x and y. Thus we obtain

$$
\begin{aligned}
(3.3)\quad z = {}&\tfrac{1}{2}y^2 + \tfrac{1}{6}uy^3 + \tfrac{1}{6}xy^3 + \tfrac{1}{6}u^2y^4 + \tfrac{1}{6}uxy^4 + \tfrac{1}{60}(10u^3 + 3c)y^5\\
&+ \tfrac{1}{8}x^2y^4 + \tfrac{1}{4}u^2xy^5 + \tfrac{1}{360}(3c' + 5d + 65u^4 + 23cu)y^6 + \cdots.
\end{aligned}
$$

We conclude that *the expansion* (3.3) *for one nonhomogeneous coördinate z as a power series in the other two nonhomogeneous coördinates x and y of a point represents an integral surface of equations* (2.3) *in a sufficiently small neighborhood of an ordinary point x whose local nonhomogeneous coördinates are* 0, 0, 0, *the vertices of the local tetrahedron of reference being the points x, x_u, x_v, x_{vv} and the unit point being suitably chosen.*

4. Contact of a quadric surface with a developable surface. Before taking up the proof of Theorem I, it is of some interest to discuss quadric surfaces. Writing the equation of the most general quadric surface and demanding that it be satisfied by the series (3.3) identically in x and y as far as terms of the second degree, we find that there exists a three-parameter family of quadric surfaces with second-order contact. The equation of a general one of these is

$$(4.1)\qquad z - \tfrac{1}{2}y^2 + k_2xz + k_3yz + k_4z^2 = 0,$$

where k_2, k_3 and k_4 are the parameters. The quadric surface has third-order contact if, and only if, $k_2 = 0$, $k_3 = -u/3$. We conclude that there exists a one-parameter family of quadric surfaces having third-order contact. The equation of a general one of these is

$$(4.2)\qquad z - \tfrac{1}{2}y^2 - uyz/3 + k_4z^2 = 0,$$

where k_4 is the parameter. It is interesting to note that the quadrics (4.2) are all quadric cones with their vertex at the focal point on the generator through the origin. Finally, we find that the quadric (4.2) never has fourth-order contact for any value of k_4.

5. Contact of a cubic surface with a developable surface. We now proceed to the proof of Theorem I. We write the equation of the most general cubic surface and demand that it be satisfied by the power series (3.3) identically in x and y as far as terms of the fifth degree. Thus we find that each member of the three-parameter family of cubic surfaces whose equations are

$$\begin{aligned}(5.1)\qquad &(9k_1 + 15k_3)y^3 + k_4z^3 + (12k_1 + 40uk_3)xz^2 + 30k_3xyz \\ &+ [2uk_2 + 8u^2k_1 + (40u^3 + 18c)k_3]yz^2 - 6k_2z^2 - 18k_1yz \\ &+ (6uk_1 + 3k_2 + 30u^2k_3)y^2z - 90k_3z + 45k_3y^2 = 0,\end{aligned}$$

has fifth-order contact with S. Moreover, there is a unique member of this family which has sixth-order contact with S. Its equation is

$$(5.2)\qquad 135y^3 + (80u^3 + 108c)z^3 + 180xz^2 + 120u^2yz^2 + 90uy^2z - 270yz = 0.$$

This cubic surface could be called the *osculating cubic surface* of the developable surface. Finally, the cubic surface (5.2) never has contact of the seventh order. These remarks complete the proof of Theorem I.

We have incidentally proved that there are no cubic surfaces which are also tangent developables of twisted curves.

6. The degenerate developables. Let us now consider the cases that arise when the developable surface S is a plane or a cone. In the first case we may assume that the equation of the plane is $z = 0$. Then it is easy to see that there exists a thirteen-parameter family of cubic surfaces with second-order contact and no non-composite cubic surface with contact of the third order.

If the developable surface S is a cone and P is a point on S at which we wish to investigate the order of contact of cubic surfaces, we may suppose that the director curve $y(v)$ is a plane curve through P and that the proportionality factor and variable v have been chosen so that $y(v)$ is a solution of the Laguerre-Forsythe canonical differential equation

$$(6.1)\qquad y''' = py.$$

Let $z = 0$ be the plane of this curve and pick the points y, y', $\frac{1}{2}y''$ as three of the vertices of a local tetrahedron of reference. The fourth vertex of the tetrahedron is picked as the vertex a of the cone and the unit point is selected so that a point $X = x_1y + x_2y' + \frac{1}{2}x_3y'' + x_4a$ will have local homogeneous coördinates proportional to (x_1, x_2, x_3, x_4). Then the power series expansion for the cone analogous to (3.3) for a general developable is

(6.2) $$y = x^2 + px^5/10 + p'x^6/60 + \cdots.$$

To see that this is true we need only remark that with the coördinate system chosen as it is the equation of the cone cannot contain the variable z and must be the same as the equation of the director curve $y(v)$ in the plane $z = 0$. The expansion (6.2) is well known to be the latter equation (see Lane [2, p. 57]) when the curve $y(v)$ satisfies the differential equation (6.1) and the coördinate system is selected as it is.

Employing this expansion we readily find that there exists a two-parameter family of cubic surfaces with sixth-order contact with S at P and that all these cubic surfaces are actually cubic cones with the same vertex as S. We conclude that the theory of contact of a cubic surface with a cone is coextensive with the theory of contact of a plane cubic curve with a plane curve, at least for orders of contact greater than five. It follows from this latter theory that there exists a one-parameter family of cubic surfaces (actually cubic cones) with seventh-order contact, and that ordinarily there is a unique osculating cubic cone with eighth-order contact. If at every point of the cone the osculating cubic cone has ninth-order contact, then the original cone is itself a cubic cone.

II. Nondevelopable Ruled Surfaces.

7. The differential equations. As is well known, the parametric vector equation of a nondevelopable ruled surface S can be written in the form

$$x(u, v) = y(v) + uz(v),$$

where $y(v)$ and $z(v)$ define director curves on S which satisfy differential equations of the form (see Lane [2, p. 165])

(7.1) $$y'' = -\tfrac{1}{2}B(v)y + C(v)z, \qquad z'' = -A(v)y + \tfrac{1}{2}B(v)z.$$

It follows that $x(u, v)$ satisfies the completely integrable system of partial differential equations

(7.2) $$x_{uu} = 0, \qquad x_{vv} = -\tfrac{1}{2}\gamma_u x + \gamma x_u,$$

where γ is defined by

(7.3) $$\gamma = Au^2 + Bu + C.$$

Conversely, if (7.2) is a completely integrable system, then γ is of the form (7.3) and $x = y + uz$, where y and z satisfy the differential equations (7.1).

If $\gamma \equiv 0$, the integral surfaces of (7.3) are quadrics, while if $\gamma \not\equiv 0$, the conditional equation $\gamma = 0$ defines the flecnode curves on S. We shall hereafter suppose that $\gamma \neq 0$ on the region of the surface in which we are interested.

The most general transformation of asymptotic parameters u, v and proportionality factor which leaves invariant the form of (7.2) is given by

$$u^* = U(u), \qquad v^* = V(v), \qquad x = \tau x^*, \qquad \tau U'V' \neq 0, \tag{7.4}$$

where τ, U and V are defined by

$$\begin{gathered} \tau = (au + b)(cv + d), \qquad U = (eu + f)/(au + b), \\ V = (gv + h)/(cv + d), \end{gathered} \tag{7.5}$$

the quantities a, b, c, d, e, f, g and h being constants such that

$$(eb - af)(gd - hc) \neq 0.$$

Under this transformation γ is carried into

$$\gamma^* = U'\gamma/V'^2 = A^*u^{*2} + B^*u^* + C^*,$$

where A^*, B^* and C^* are defined by

$$\begin{aligned} A^* &= (gd - hc)^2(Ab^2 - Bba + Ca^2)/(eb - af)(cv^* - g)^4, \\ B^* &= -(gd - hc)^2(2Abf - Baf - Bbe + 2Cae)/(eb - af)(cv^* - g)^4, \\ C^* &= (gd - hc)^2(Af^2 - Bfe + Ce^2)/(eb - af)(cv^* - g)^4. \end{aligned} \tag{7.6}$$

8. The power series expansions. Let P_x be an ordinary point on S. Since S is not developable, the points x, x_u, x_v, x_{uv} may be chosen as the vertices of a local tetrahedron of reference with unit point so selected that a point $X = x_1x + x_2x_u + x_3x_v + x_4x_{uv}$ will have local homogeneous coördinates proportional to (x_1, x_2, x_3, x_4). If $X = x(u + \Delta u, v + \Delta v)$ is a point on S near x, we can employ a Taylor series and the differential equations (7.2) in a familiar fashion to express each of the local coördinates of X as a power series in Δu and Δv. These series, to as many terms as will be needed in the subsequent discussion, are as follows:

$$\begin{aligned} x_1 &= 1 - \tfrac{1}{4}\gamma_u\Delta v^2 - \tfrac{1}{4}\gamma_{uu}\Delta u\Delta v^2 - \tfrac{1}{12}\gamma_{uv}\Delta v^3 - \tfrac{1}{12}\gamma_{uuv}\Delta u\Delta v^3 \\ &\quad + \tfrac{1}{96}(\gamma_u^2 - 2\gamma\gamma_{uu} - 2\gamma_{uvv})\Delta v^4 + \cdots, \\ x_2 &= \Delta u + \tfrac{1}{2}\gamma\Delta v^2 + \tfrac{1}{4}\gamma_u\Delta u\Delta v^2 + \tfrac{1}{6}\gamma_v\Delta v^3 + \tfrac{1}{12}\gamma_{uv}\Delta u\Delta v^3 \\ &\quad + \tfrac{1}{24}\gamma_{vv}\Delta v^4 + \tfrac{1}{96}(\gamma_u^2 - 2\gamma\gamma_{uu} + 2\gamma_{uvv})\Delta u\Delta v^4 \\ &\quad + \tfrac{1}{120}(\gamma_u\gamma_v - \gamma\gamma_{uv} + \gamma_{vvv})\Delta v^5 + \cdots, \end{aligned}$$

$$
\begin{aligned}
(8.1)\qquad x_3 &= \Delta v - \tfrac{1}{12}\gamma_u\Delta v^3 - \tfrac{1}{12}\gamma_{uu}\Delta u\Delta v^3 - \tfrac{1}{24}\gamma_{uv}\Delta v^4 \\
&\quad - \tfrac{1}{24}\gamma_{uuv}\Delta u\Delta v^4 + \tfrac{1}{480}(\gamma_u{}^2 - 2\gamma\gamma_{uu} - 6\gamma_{uvv})\Delta v^5 + \cdots, \\
x_4 &= \Delta u\Delta v + \tfrac{1}{6}\gamma\Delta v^3 + \tfrac{1}{12}\gamma_u\Delta u\Delta v^3 + \tfrac{1}{12}\gamma_v\Delta v^4 + \tfrac{1}{24}\gamma_{uv}\Delta u\Delta v^4 \\
&\quad + \tfrac{1}{40}\gamma_{vv}\Delta v^5 + \tfrac{1}{480}(\gamma_u{}^2 - 2\gamma\gamma_{uu} + 6\gamma_{uvv})\Delta u\Delta v^5 \\
&\quad + \tfrac{1}{720}(\gamma_u\gamma_v - \gamma\gamma_{uv} + 4\gamma_{vvv})\Delta v^6 + \cdots.
\end{aligned}
$$

If nonhomogeneous coördinates are introduced by placing

$$x = x_2/x_1, \qquad y = x_3/x_1, \qquad z = x_4/x_1,$$

we find that x, y and z may be expressed in power series as follows:

$$
\begin{aligned}
x &= \Delta u + \tfrac{1}{2}\gamma\Delta v^2 + \tfrac{1}{2}\gamma_u\Delta u\Delta v^2 + \tfrac{1}{6}\gamma_v\Delta v^3 + \tfrac{1}{4}\gamma_{uu}\Delta u^2\Delta v^2 \\
&\quad + \tfrac{1}{6}\gamma_{uv}\Delta u\Delta v^3 + \tfrac{1}{24}(\gamma_{vv} + 3\gamma\gamma_u)\Delta v^4 + \tfrac{1}{12}\gamma_{uuv}\Delta u^2\Delta v^3 \\
&\quad + \tfrac{1}{24}(\gamma_{uvv} + 3\gamma\gamma_{uu} + 3\gamma_u{}^2)\Delta u\Delta v^4 \\
&\quad + \tfrac{1}{120}(\gamma_{vvv} + 4\gamma\gamma_{uv} + 6\gamma_u\gamma_v)\Delta v^5 + \cdots, \\
(8.2)\qquad y &= \Delta v + \tfrac{1}{6}\gamma_u\Delta v^3 + \tfrac{1}{6}\gamma_{uu}\Delta u\Delta v^3 + \tfrac{1}{24}\gamma_{uv}\Delta v^4 + \tfrac{1}{24}\gamma_{uuv}\Delta u\Delta v^4 \\
&\quad + \tfrac{1}{120}(4\gamma_u{}^2 + 2\gamma\gamma_{uu} + \gamma_{uvv})\Delta v^5 + \cdots, \\
z &= \Delta u\Delta v + \tfrac{1}{6}\gamma\Delta v^3 + \tfrac{1}{3}\gamma_u\Delta u\Delta v^3 + \tfrac{1}{12}\gamma_v\Delta v^4 + \tfrac{1}{4}\gamma_{uu}\Delta u^2\Delta v^3 \\
&\quad + \tfrac{1}{8}\gamma_{uv}\Delta u\Delta v^4 + \tfrac{1}{120}(3\gamma_{vv} + 5\gamma\gamma_u)\Delta v^5 + \tfrac{1}{12}\gamma_{uuv}\Delta u^2\Delta v^4 \\
&\quad + \tfrac{1}{120}(9\gamma_u{}^2 + 7\gamma\gamma_{uu} + 4\gamma_{uvv})\Delta u\Delta v^5 \\
&\quad + \tfrac{1}{720}(16\gamma_u\gamma_v + 9\gamma\gamma_{uv} + 4\gamma_{vvv})\Delta v^6 + \cdots.
\end{aligned}
$$

It is now possible to calculate an expansion for z as a power series in x and y. Thus we obtain

$$
\begin{aligned}
(8.3)\qquad z &= xy - \tfrac{1}{3}\gamma y^3 - \tfrac{1}{3}\gamma_u xy^3 - \tfrac{1}{12}\gamma_v y^4 - \tfrac{1}{6}\gamma_{uu}x^2y^3 - \tfrac{1}{12}\gamma_{uv}xy^4 \\
&\quad + \tfrac{1}{60}(10\gamma\gamma_u - \gamma_{vv})y^5 - \tfrac{1}{24}\gamma_{uuv}x^2y^4 \\
&\quad + \tfrac{1}{60}(10\gamma\gamma_{uu} + 10\gamma_u{}^2 - \gamma_{uvv})xy^5 \\
&\quad + \tfrac{1}{360}(20\gamma_u\gamma_v + 15\gamma\gamma_{uv} - \gamma_{vvv})y^6 + \cdots.
\end{aligned}
$$

The expansion (8.3) *for one nonhomogeneous coördinate z as a power series in the other two nonhomogeneous coördinates x and y of a point represents an integral surface of equations* (7.2) *in a sufficiently small neighborhood of an ordinary point x whose local nonhomogeneous coördinates are* 0, 0, 0, *the vertices of the local tetrahedron of reference being the points x, x_u, x_v, x_{uv} and the unit point being suitably chosen.* We remark that all the terms in the series (8.1), (8.2) and (8.3) except those of the highest degree can be obtained from more general series to be found in the treatise of Lane [2, pp. 127-129]. In fact we merely have to put $p = \theta_u = \theta_v = \beta = 0$, $q = -\tfrac{1}{2}\gamma_u$ in the series (V: 4.3), (V: 4.6) and (V: 4.7) of his book in order to obtain these terms.

9. Contact of a cubic surface with a nondevelopable ruled surface. We now proceed to the proof of Theorem II. We write down the equation of the most general cubic surface and demand that it be satisfied by the power series (8.3) identically in x and y as far as terms of the fifth degree. Thus we find that the equation of any cubic surface having fifth-order contact with S may be written as

$$(9.1)\quad (1/3)\gamma y^3 + Dz^3 + (z - xy)(Px - \tfrac{1}{4}(\gamma_v/\gamma)y + Mz + 1) + yz(Iy + Jz) =$$

where D is an arbitrary parameter and P, M, I and J are defined by the following formulas:

$$\begin{aligned} P &= (15\gamma_v^2 + 40\gamma^2\gamma_u - 12\gamma\gamma_{vv})/80\gamma^3, \\ M &= (40\gamma^2\gamma_u\gamma_v + 12\gamma\gamma_v\gamma_{vv} - 80\gamma^3\gamma_{uv} - 15\gamma_v^3)/320\gamma^4, \\ I &= (5\gamma_v^2 + 40\gamma^2\gamma_u - 4\gamma\gamma_{vv})/80\gamma^2, \\ J &= (15\gamma_u\gamma_v^2 + 40\gamma^2\gamma_u^2 - 12\gamma\gamma_u\gamma_{vv} + 40\gamma^3\gamma_{uu})/240\gamma^3. \end{aligned}$$

If we examine the terms of the sixth degree we see that three of the directions of the sextuple point which the curve of intersection of the surface S and the cubic surface (9.1) has at P_x coincide in the direction $dv = 0$, i. e., the direction of the generator through P_x, and the other three satisfy the equation

$$(D - P\gamma_{uu}/6)du^3 + \mathfrak{A}du^2dv + \mathfrak{B}\,dudv^2 + \mathfrak{C}\,dv^3 = 0,$$

where $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ are defined by the following formulas:

$$\begin{aligned} \mathfrak{A} = {} & [(15\gamma_v^2 - 12\gamma\gamma_{vv} - 40\gamma^2\gamma_u)(\gamma_u\gamma_v - \gamma\gamma_{uv}) \\ (9.2)\qquad & + 40\gamma^3(\gamma_u\gamma_v - \gamma\gamma_{uv})_u]/960\gamma^4, \\ \mathfrak{B} = {} & [1600\gamma^4(2\gamma\gamma_{uu} - \gamma_u^2) + 960\gamma^3(\gamma_u\gamma_v - \gamma\gamma_{uv})_v \\ (9.3)\qquad & - 2400\gamma^2\gamma_v(\gamma_u\gamma_v - \gamma\gamma_{uv}) + (12\gamma\gamma_{vv} - 15\gamma_v^2)^2]/57600\gamma^4, \\ \mathfrak{C} = {} & [-20\gamma^2(\gamma_u\gamma_v - \gamma\gamma_{uv}) - 4\gamma^2\gamma_{vvv} + 18\gamma\gamma_v\gamma_{vv} - 15\gamma_v^3]/1440\gamma^2. \end{aligned}$$

We conclude that there exists a unique cubic surface with fifth-order contact and intersecting S in a curve with a sextuple point, four of whose directions coincide in the direction of the generator. The remaining two satisfy

$$\mathfrak{A}du^2 + \mathfrak{B}\,dudv + \mathfrak{C}\,dv^2 = 0.$$

It is obvious that there exists a cubic surface having sixth-order contact with S if and only if $\mathfrak{A} = \mathfrak{B} = \mathfrak{C} = 0$ and that then it is unique. In particular this will be true at every point of S if S is itself a cubic surface. To complete the proof of Theorem II, it is sufficient to prove the converse of this last statement.

Suppose now that $\mathfrak{A} \equiv \mathfrak{B} \equiv \mathfrak{C} \equiv 0$. From (7. 3) we find that

$$\gamma_u\gamma_v - \gamma\gamma_{uv} = \pi u^2 - 2\rho u + \sigma,$$

where

$$\pi = AB' - A'B, \qquad \rho = A'C - AC', \qquad \sigma = BC' - B'C.$$

Then we find that $960\gamma^4\mathfrak{A}$ is a polynomial of the sixth degree in u whose leading coefficient is $(\ldots)\pi + 80A^5\rho$ and in which the coefficient of u^5 is $(\ldots)\pi + (\ldots)\rho - 80A^3\sigma$. Since $\mathfrak{A} \equiv 0$, these coefficients must vanish. If $\pi \equiv 0$, then either $A \equiv 0$ or $\rho \equiv 0$. In either case $\rho \equiv A'C - AC' \equiv 0$. Then either $A \equiv 0$ or $\sigma \equiv 0$. If $A \equiv 0$, we find from (9. 2) that $960\gamma^4\mathfrak{A}$ is a quadratic polynomial in u whose coefficients are, respectively, given by

$$\begin{aligned} L_1 &= (-40B^3 - 12BB'' + 15B'^2)\sigma, \\ M_1 &= -(80B^2C + 12BC'' + 12B''C - 30B'C')\sigma, \\ N_1 &= (-40BC^2 - 12CC'' + 15C'^2)\sigma. \end{aligned}$$

Since $\mathfrak{A} \equiv 0$, these coefficients must vanish. Therefore

$$0 = C^2L_1 - BCM_1 + B^2N_1 = 15\sigma^3.$$

We conclude that $\pi \equiv 0$ implies $\rho \equiv \sigma \equiv 0$, $\gamma_u\gamma_v - \gamma\gamma_{uv} \equiv 0$. If the last of these equations holds, it is easy to show that

$$\gamma = f(u)g(v) = (lu^2 + mu + n)g(v), \tag{9. 5}$$

where l, m and n are constants not all zero and $g(v) \neq 0$.

The equation $\mathfrak{A} = 0$ may be regarded as a linear homogeneous differential equation satisfied by $\eta(u, v) \equiv \gamma_u\gamma_v - \gamma\gamma_{uv}$ with independent variable u. Uniqueness theorems for such differential equations imply that if there exists a point (u_0, v_0) for which $\eta(u_0, v_0) = 0$, then $\eta(u, v_0) \equiv 0$ in u. Suppose that there is a value v_0 for which $\pi(v_0) \neq 0$. We put

$$u_0 = \{\rho(v_0) + [\rho^2(v_0) - \pi(v_0)\sigma(v_0)]^{\frac{1}{2}}\}/\pi(v_0).$$

Then $\eta(u, v_0) \equiv 0$ and this is a contradiction. Hence $\pi \equiv 0$, so that $\rho \equiv \sigma \equiv 0$ by our proof above. Thus we have proved

LEMMA 9. 1. *The equations $\gamma_{uuu} = 0$, $\mathfrak{A} = 0$ hold if and only if γ is given by* (9. 5).

If $\mathfrak{A} \equiv 0$ we can reduce the equation $\mathfrak{B} \equiv 0$ by means of (9. 5) to the form

$$\zeta \equiv 12gg'' - 15g'^2 + 40\delta g^3 = 0, \qquad \delta^2 = m^2 - 4ln, \tag{9. 6}$$

and we can reduce the equation $\mathcal{C} \equiv 0$ to the form

$$(9.7) \qquad \theta \equiv -4g^2g''' + 18gg'g'' - 15g'^3 = 8g^{7/2}(g^{-\frac{1}{2}})''' = 0.$$

Then the identity $g\zeta' - 3g'\zeta \equiv -3\theta$ proves the following

LEMMA 9.2. *If $\gamma_{uuu} = \mathcal{A} = \mathcal{B} = 0$, then $\mathcal{C} = 0$.*

If $\mathcal{C} \equiv 0$, it follows from (9.7) that g must have the form

$$g(v) = 1/(\lambda v^2 + \mu v + \nu)^2,$$

where λ, μ and ν are constants not all zero. Such a function will satisfy (9.6) if and only if

$$(9.8) \qquad 3\epsilon^2 = -10\delta, \qquad \epsilon^2 = \mu^2 - 4\lambda\nu, \qquad \delta^2 = m^2 - 4ln.$$

Therefore, γ satisfies $\gamma_{uuu} = \mathcal{A} = \mathcal{B} = \mathcal{C} = 0$ if and only if

$$(9.9) \qquad \gamma = (lu^2 + mu + n)/(\lambda v^2 + \mu v + \nu)^2,$$

where the constants satisfy (9.8).

We are now in a position to complete the proof of Theorem II by showing that any integral surface of (7.2) with γ defined by (9.9) is necessarily a cubic surface.

If $\delta = 0$, so that $\epsilon = 0$, we make the transformation (7.4), (7.5) selecting a, b so that $lb^2 - mba + na^2 = 0$. Then $A^* = 0$. Since the equation $B^2 - 4AC = 0$ is invariant, it follows that $B^* = 0$. We choose e, f so that $be - af = 1$. Since $\epsilon = 0$, if we choose c, d so that $\lambda d^2 - \mu dc + \nu c^2 = 0$, then also $2\lambda dh - \mu dg - \mu ch + 2\nu cg = 0$. Then we find that

$$C^* = (gd - hc)^2(lf^2 - mfe + ne^2)/(\lambda h^2 - \mu gh + \nu g^2)^2 \neq 0.$$

Since $d = \xi\nu^{\frac{1}{2}}$, $c = \xi\lambda^{\frac{1}{2}}$, where ξ is arbitrary, we can make $C^* = 3$ by picking $\xi = 3^{\frac{1}{2}}(lf^2 - mfe + ne^2)^{-\frac{1}{2}}$, and the choice of g, h is arbitrary except that $gd - hc \neq 0$. Thus we see that we may assume that the equations (7.2) have the form

$$x_{uu} = 0, \qquad x_{vv} = 3x_u.$$

Four linearly independent solutions of these equations are

$$x_1 = 1, \qquad x_2 = v, \qquad x_3 = u + 3v^2/2, \qquad x_4 = uv + \tfrac{1}{2}v^3.$$

These are the parametric equations of a cubic surface known as Cayley's cubic scroll (see Lane [2, p. 136] or Wilczynski [3, p. 145]). Its implicit equation is $x_1(x_1x_4 - x_2x_3) + x_2^3 = 0$.

If $\delta \neq 0$, so that $\epsilon \neq 0$, we make the transformation (7.4), (7.5) selecting a, b, c, d, e, f, g and h so that

$$
\begin{gathered}
lb^2 - mba + na^2 = lf^2 - mfe + ne^2 = 0, \\
\lambda d^2 - \mu dc + \nu c^2 = \lambda h^2 - \mu hg + \nu g^2 = 0, \\
(be - af)(gd - hc) \neq 0.
\end{gathered} \tag{9.10}
$$

Then $A^* = C^* = 0$ and B^* is given by

$$B^* = -\frac{(gd - hc)^2(2lbf - maf - mbe + 2nae)}{v^{*2}(be - af)(2\lambda dh - \mu dg - \mu ch + 2\nu cg)^2}.$$

Making use of equations (9.10) we find that $B^* = \delta/\epsilon^2 v^{*2}$. Then from (9.8) we conclude that $B^* = -3/10v^{*2}$. Therefore, we may assume that the equations (7.2) have the form

$$x_{uu} = 0, \qquad x_{vv} = -3x/20v^2 + 3ux_u/10v^2.$$

Four linearly independent solutions of these equations are given by

$$x_1 = v^{1-r}, \qquad x_2 = v^r, \qquad x_3 = v^{1-s}u, \qquad x_4 = v^s u,$$

where $r = \frac{1}{2} + 10^{-\frac{1}{2}}$, $s = \frac{1}{2} + 2 \cdot 10^{-\frac{1}{2}}$. This is also a cubic surface and its implicit equation is $x_1{}^2 x_4 - x_2{}^2 x_3 = 0$. For a discussion of this cubic surface, see Wilczynski [3, p. 145].

These remarks complete the proof of Theorem II and also prove the known

COROLLARY. *Besides the cubic cones, there are only two projectively inequivalent cubic ruled surfaces.*

10. Geometrical interpretations of the invariants $\mathcal{A}$ and $\mathcal{B}$. We have shown that $\mathcal{A} = 0$ if, and only if, $\gamma_u \gamma_v - \gamma \gamma_{uv} = 0$. It then follows from some results stated in Lane [2, pp. 165-166] that $\mathcal{A} = 0$ if, and only if, the flecnode curves are asymptotic curves, or if, and only if, S possesses two (distinct or coincident) straight line director curves. These are the flecnode curves of the preceding characterization. Moreover, $\mathcal{A} = 0$ if, and only if, S belongs to a linear congruence and the directrices of the congruence are the flecnode curves above. Finally, $\mathcal{A} = 0$ if, and only if, the curved asymptotics belong to linear complexes.

To interpret $\mathcal{B}$ we consider the curve of intersection of S with its tangent plane at the point P_x. The intersection has two branches, one of which is the generator $y = z = 0$ and the other of which may be obtained by setting $z = 0$ in the power series (8.3) and solving for x as a power series in y. Thus we find that the equations of this branch are

$$x = \tfrac{1}{3}\gamma y^2 + \tfrac{1}{12}\gamma_v y^3 + \tfrac{1}{180}(3\gamma_{vv} - 10\gamma\gamma_u)y^4 - \tfrac{1}{360}(10\gamma_u\gamma_v + 5\gamma\gamma_{uv} - \gamma_{vvv})y^5 + \cdots, \quad z = 0. \tag{10.1}$$

The equation of the osculating conic of the curve (10.1) is found to be

$$z = Px^2 + x - \tfrac{1}{3}\gamma y^2 - \tfrac{1}{4}(\gamma_v/\gamma)xy = 0. \tag{10.2}$$

We see from (9.1) that this conic may also be given the following characterization. Any cubic surface with fifth-order contact with S intersects the tangent plane in a cubic curve which decomposes into the generator $y = 0$ and the conic (10.2). Our desired interpretation of $\mathcal{C}$ is supplied by the fact that the conic (10.2) hyperosculates the curve (10.1) if, and only if, $\mathcal{C} = 0$.

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A LIPSCHITZ CONDITION PRESERVING EXTENSION FOR A VECTOR FUNCTION.*

By F. A. Valentine.

1. Introduction. The existence of an extension of the range of definition of a function $f(x)$, defined on a metric space $\mathfrak{M}$ to a metric space $\mathfrak{M}'$, so as to preserve a Lipschitz condition (1), depends upon $\mathfrak{M}$ and $\mathfrak{M}'$. In a previous paper [5], the author has established the extension when $\mathfrak{M}$ and $\mathfrak{M}'$ are both Euclidean planes. Both McShane [3], and the author [5], have established, in different ways, the extension when $\mathfrak{M}$ is a metric space, and when $\mathfrak{M}'$ is the one-dimensional real space. Zorn [6], has shown that if $\mathfrak{M}$ and $\mathfrak{M}'$ are a large class of Minkowski spaces, the extension, in general, does not exist. For another example, the result of a discussion with my colleague W. T. Puckett, see Valentine, [5]. *In this paper it is shown that the extension exists when each $\mathfrak{M}$ and $\mathfrak{M}'$ is,* (1) *the n-dimensional Euclidean space*; (2) *the surface of the n-dimensional Euclidean sphere*; or (3) *the general Hilbert space.* The author wishes to take this opportunity to express again his gratitude to his colleagues Max Zorn and W. T. Puckett who have made helpful suggestions concerning these topics.

The following notation is used. A sphere S_i in $\mathfrak{M}$ with radius r_i and center x_i is the set of points x for which $\| x, x_i \| \leqq r_i$. Any notation in $\mathfrak{M}'$ corresponding to one in $\mathfrak{M}$ will be distinguished by means of a prime. The extension of $f(x)$ so as to preserve a Lipschitz condition

$$\| f(x_1), f(x_2) \|' \leqq \| x_1, x_2 \| \tag{1}$$

necessarily depends upon the following property.

Property E. *Consider two sets of spheres M and M' contained in the metric spaces $\mathfrak{M}$ and $\mathfrak{M}'$ respectively. Suppose that to each sphere $S_i \varepsilon M$, having center x_i and radius r_i, there corresponds a sphere $S'_i \varepsilon M'$, having center x'_i and radius r'_i. Furthermore suppose that*

$$r_i = r'_i, \qquad \| x'_i, x'_j \|' \leqq \| x_i, x_j \|, \tag{2}$$

* Received July 28, 1943; Presented to the American Mathematical Society, March 17, 1943.

for all corresponding spheres S_i and S'_i, and for all corresponding pairs (S_i, S_j) and (S'_i, S'_j).

The spaces $\mathfrak{M}$ and $\mathfrak{M}'$ are said to have the extensibility property E *if under the above assumptions the condition*

$$\prod_M S_i \neq 0 \tag{3}$$

implies

$$\prod_{M'} S'_i \neq 0. \tag{4}$$

In establishing the validity of Property E for the case $\mathfrak{M} = \mathfrak{M}' = \mathfrak{R}_n$, the n-dimensional Euclidean space, it is first proved for the range $(i = 1, \cdots, n + 1)$. The property E is then proved for arbitrary sets M and M' by means of a theorem of Helly, [2]. An advantage of the above procedure lies in the fact that property E is a *necessary* as well as *sufficient* condition for the extensibility of $f(x)$ so as to be Lipschitz preserving.

2. Property E in $\mathfrak{R}_n$.

THEOREM 1. *Property* E *holds when each $\mathfrak{M}$ and $\mathfrak{M}'$ is the n-dimensional Euclidean space $\mathfrak{R}_n$.*

Proof. Let S_i $(i = 1, \cdots, n + 1)$ be *any* set of $n + 1$ spheres in the set M. We shall first prove that condition (3) implies condition (4) for $(i = 1, \cdots, n + 1)$. Let $\Delta(x'_1, \cdots, x'_{n+1})$ be the Euclidean simplex (degenerate or non-degenerate) determined by the centers x'_i of S'_i $(i = 1, \cdots, n + 1)$. *Suppose that $\Delta(x'_1, \cdots, x'_{n+1})$ is not covered by the sets S'_i.* Then choose x and x' so that

$$x \,\varepsilon \prod_{i=1}^{n+1} S_i, \qquad x' \,\varepsilon\, \Delta(x'_1, \cdots, x'_{n+1}) - \sum_{i=1}^{n+1} S'_i, \tag{5}$$

and let R_i and R'_i be the n-dimensional vectors drawn from x and x' to x_i and x'_i respectively. Conditions (5) and (2) imply that

$$R'_i \cdot R'_i > R_i \cdot R_i, \qquad (i \text{ not summed}), \tag{6}$$

$$(R'_i - R'_j) \cdot (R'_i - R'_j) \leqq (R_i - R_j) \cdot (R_i - R_j), \tag{7}$$

where the products involved are vector dot products. Conditions (6) and (7) yield the inequalities

$$R'_i \cdot R'_j > R_i \cdot R_j. \tag{8}$$

Since the point x' is contained in $\Delta(x'_1, \cdots, x'_{n+1})$, there exist constants a'_i $(i = 1, \cdots, n+1)$ such that

$$(9) \qquad a'_i \geqq 0, \qquad \sum_{i=1}^{n+1} a'_i \neq 0, \qquad a'_i R'_i = 0, \qquad (i \text{ summed}).$$

Multiplying (8) by $a'_i a'_j$, summing on i and j, one obtains

$$(10) \qquad (a'_i R'_i) \cdot (a'_j R'_j) > (a'_i R_i) \cdot (a'_j R_j),$$

which by (9) implies that

$$| a'_i R_i |^2 < 0, \qquad (i \text{ summed}),$$

which is impossible. Hence $\Delta(x'_1, \cdots, x'_{n+1})$ is covered by the sets S'_i. Furthermore conditions (3) imply that

$$(11) \qquad S_i \cdot S_j \cdot B(x_i, x_j) \neq 0,$$

where $B(x_i, x_j)$ is the side of the simplex Δ joining x_i and x_j. Conditions (2) and (11) then yield the conditions

$$(12) \qquad S'_i \cdot S'_j \cdot B(x'_i, x'_j) \neq 0.$$

By a theorem of Knaster, Kuratowski and Mazurkiewicz,[1] conditions (12) and the fact that $\Delta(x'_1, \cdots, x'_{n+1})$ is covered by the sets S'_i, imply that $\prod_{i=1}^{n+1} S'_i \neq 0$.

Since we have shown that *each* set of $n + 1$ of the spheres S'_i have a point in common, *it follows by a theorem of Helly,*[2] *that all the spheres S'_i have a point in common.* Thus Theorem 1 is proved.

3. Property E **in $\mathcal{K}_n$.** Let $\mathcal{K}_n$ be the surface of the $(n+1)$-dimensional Euclidean sphere E_{n+1} in $\mathcal{R}_{n+1}$. Although Lemmas 1-3 are used to establish property E for $\mathcal{K}_n$, they are of independent geometric interest.

LEMMA 1. *Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two n-dimensional spherical surfaces $\mathcal{K}_n$ and $\mathcal{K}'_n$ belonging to two Euclidean spheres E_{n+1} and E'_{n+1}, respectively,*

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[1] See Alexandroff and Hopf, [1, p. 377]. The theorem states: *If the closed sets $A_0, \cdots, A_n$ cover the simplex T, and if for each side $a_{i_0}, \cdots, a_{i_T}$ of T, we have*

$$a_{i_0}, \cdots, a_{i_T} \subset A_{i_0} + \cdots + A_{i_T},$$

then $A_0 \cdot A_1 \cdot \ldots \cdot A_n \neq 0$.

[2] See [2] or [1, p. 297]. The theorem states: *If each $n + 1$ sets of a family of closed bounded convex sets of the n-dimensional Euclidean space intersect, then there is a point in common to all the sets.*

of equal radii; if in the statement of property E, $i = 1, \cdots, n+1$, *then conditions* (2) *and* (3) *imply conditions* (4).

Proof. Since $i = 1, \cdots, n+1$, *the set of centers x_i of S_i must lie on the surface of a closed hemisphere H_{n+1} of the sphere E_{n+1}.* A corresponding statement holds for the points x'_i. It should be remembered that S_i and S'_i are here n-dimensional spheres in $\mathcal{K}_n$ and $\mathcal{K}'_n$ respectively. We *first assume* that x'_i are interior points of the surface of H'_{n+1}. Denote the surface simplex (degenerate or non-degenerate) on H_{n+1}, determined by the points x_i, by $\sigma(x_1, \cdots, x_{n+1})$. Attach a corresponding meaning to $\sigma(x'_1, \cdots, x'_{n+1})$. Condition (3) implies that

$$S_i \cdot S_j \cdot B(x_i, x_j) \neq 0, \qquad (i, j = 1, \cdots, n+1), \tag{13}$$

where $B(x_i, x_j)$ is the side of the simplex σ joining x_i and x_j.

By virtue of conditions (2) and (13) we have $\| x'_i, x'_j \| \leqq r'_i + r'_j$, so that

$$S'_i \cdot S'_j \cdot B(x'_i, x'_j) \neq 0. \tag{14}$$

Choose the point x so that

$$x \,\varepsilon \prod_{i=1}^{n+1} S_i \neq 0. \tag{15}$$

Suppose that the simplex $\sigma(x'_1, \cdots, x'_{n+1})$ is *not* covered by the spheres S'_i. Hence choose $x' \,\varepsilon\, \sigma(x'_1, \cdots, x'_{n+1}) - \sum_{i=1}^{n+1} S'_i$, which implies that

$$\| x', x'_i \|' > \| x, x_i \| = r_i = r'_i. \tag{16}$$

We now show that (16) contradicts the hypotheses (2). Let R_i, R be the $(n+1)$*-dimensional vectors in $\mathcal{R}_{n+1}$ drawn from the center of the sphere E_{n+1} to the points x_i and x, respectively. Attach a corresponding meaning to* R'_i, R'. Conditions (2) imply that

$$(R'_i - R'_j) \cdot (R'_i - R'_j) \leqq (R_i - R_j) \cdot (R_i - R_j) \tag{17}$$

hold, where these are vector dot products. Furthermore conditions (16) yield the inequalities

$$(R' - R'_i) \cdot (R' - R'_i) > (R - R_i) \cdot (R - R_i). \tag{18}$$

Since $| R_i | = | R'_i | = | R | = | R' |$, conditions (17) and (18) imply that

$$R'_i \cdot R'_j \geqq R_i \cdot R_j, \tag{19}$$

$$(i, j = 1, \cdots, n+1).$$

$$R \cdot R_i > R' \cdot R'_i, \tag{20}$$

Since $x' \varepsilon \sigma(x'_1, \cdots, x'_{n+1})$, and since the x'_i are interior points of the surface of the hemisphere H'_{n+1} there exist, by a well-known property of vectors, constants λ'_i such that

$$(21) \qquad R' = \lambda'_i R'_i, \qquad \lambda'_i \geqq 0, \qquad \sum_{i=1}^{n+1} \lambda'_i \neq 0, \qquad (i \text{ summed}).$$

On multiplying both sides of (19) by $\lambda'_i \lambda'_j$, summing on i and j, one obtains

$$(\lambda'_i R'_i) \cdot (\lambda'_j R'_j) \geqq (\lambda'_i R_i) \cdot (\lambda'_j R_j)$$

which, on account of (21), yields

$$(22) \qquad | \lambda'_i R_i | \leqq | R' |, \qquad (i \text{ summed}).$$

Similarly on multiplying both sides of (20) by λ'_i, summing on i, one obtains

$$R \cdot (R_i \lambda'_i) > R' \cdot R',$$

which implies, since $| R' | = | R |$, that

$$(23) \qquad | \lambda'_i R_i | > | R' |, \qquad (i \text{ summed}).$$

Since (22) and (23) are contradictory, the assumption that $\sigma(x'_1, \cdots, x'_{n+1})$ is not covered by the spheres S'_i is false. Then conditions (14) imply, by the theorem of Knaster, Kuratowski and Mazurkiewicz,[3] that $\prod_{i=1}^{n+1} S'_i \neq 0$, and Lemma 1 has been proved when the x'_i are interior points of the surface of H'_{n+1}.

Since $i = 1, \cdots, n+1$, the remaining case is that in which all of the points x'_i are on the boundary of the surface of H'_{n+1}. In this case there exist sequences of points x'_{mi}, such that $\| x'_{mi}, x'_{mj} \|' \leqq \| x_i, x_j \|$, $x'_{mi} \to x'_i$, as $m \to \infty$, and such that x'_{mi} are interior points of the surface of H'_{n+1}. For example if P is the center of the surface of H'_{n+1}, choose x'_{mi} on the great circle joining x'_i and P so that $\| x'_{mi}, x'_i \|' = \epsilon(m)$, where $\epsilon(m) \to 0$, as $m \to \infty$, for $(i = 1, \cdots, n+1)$. Then by the above proof Lemma 1 holds for the points x_i and x'_{mi}, and hence by passage to the limit, for the points x_i and x'_i.

LEMMA 2. *Let M and M' be two sets of points, each set lying on the surface $\mathcal{K}_n$ of an $(n+1)$-dimensional Euclidean sphere E_{n+1} of radius r. Suppose that to each point $x_i \varepsilon M$ there corresponds a point $x'_i \varepsilon M'$.*

[3] *Loc. cit.*[1]

Furthermore suppose that there exists in M' a set of points x'_i, $(i = 1, \cdots, m + 2, 1 \leqq m \leqq n)$ which determine an $(m + 1)$-dimensional Euclidean simplex $\Delta(x'_1, \cdots, x'_{m+2})$ containing the center O of E_{n+1} in its interior. If we have

$$\| x'_i, x'_j \| \leqq \| x_i, x_j \| \tag{24}$$

for all corresponding pairs in M and M', respectively, then

$$\| x'_i, x'_j \| = \| x_i, x_j \| \tag{25}$$

for all such pairs.

Proof. Since $\mathcal{K}'_n = \mathcal{K}_n$, we can omit the primes on the metrics entirely. The lemma is proved *first* for $x_1, \cdots, x_{m+2}$. If the strict equality in (24) holds, the lemma is empty. Hence suppose, without loss of generality, that, for example,

$$\| x'_1, x'_2 \| < \| x_1, x_2 \|. \tag{26}$$

Let (x_1, y) be a *diametrically opposite point pair* in $\mathcal{K}_n$ and attach a corresponding meaning to (x'_1, y'). Then by definition

$$\begin{aligned} \| x_1, x_i \| + \| x_i, y \| &= \pi r, \\ \| x'_1, x'_i \| + \| x'_i, y' \| &= \pi r, \end{aligned} \qquad (i = 2, \cdots, m + 2). \tag{27}$$

Conditions (24), (26) and (27) imply that

$$\begin{aligned} \| x'_i, y' \| &\geqq \| x_i, y \|, \\ \| x'_2, y' \| &> \| x_2, y \|. \end{aligned} \qquad (i = 3, \cdots, m + 2), \tag{28}$$

Let R_i, R'_i, R, R' be the $(n + 1)$-dimensional vectors drawn from the center O of E_{n+1} to the points x_i, x'_i, y, y' respectively. Since the center O of E_{n+1} is interior to the Euclidean simplex $\Delta(x'_1, \cdots, x'_{m+2})$, the point y' is *interior* to the surface simplex $\sigma(x'_2, \cdots, x'_{m+2})$, and $\sigma(x'_2, \cdots, x'_{m+2})$ must be on the interior of the surface of a hemisphere H_{n+1} of E_{n+1}. This last statement implies the existence of constants $\lambda'_i > 0$ $(i = 2, \cdots, m + 2)$, such that $R' = \lambda'_i R'_i$, (i summed). Conditions (24) yield conditions (19) with $(i, j = 2, \cdots, m + 2)$. Conditions (28) yield the inequalities,

$$\begin{aligned} R \cdot R_j &\geqq R' \cdot R'_j, \\ R \cdot R_2 &> R' \cdot R'_2. \end{aligned} \qquad (j = 3, \cdots, m + 2), \tag{29}$$

Conditions (19) and (29) imply, just as in the proof of Lemma 1, the contra-

dictory results (22) and (23) with $(i = 2, \cdots, m+2)$. Since in (26), x_1 and x_2 were chosen without restriction, Lemma 2 is proved for $(i = 1, \cdots m+2)$.

Secondly, to complete the proof, let x_s be *any* point in M, with $x_s \neq x_i$, $(i = 1, \cdots, m+2)$. Since Lemma 2 holds with $(i = 1, \cdots, m+2)$, conditions (25) imply that the $(n+1)$-dimensional Euclidean simplex $\Delta(x'_1, \cdots, x'_{m+2})$ can be moved rigidly so that $x'_i = x_i$. Hence move the set M' rigidly in $\mathcal{R}_{n+2}$ so that $x'_i = x_i$, $(i = 1, \cdots, m+2)$. *Suppose* that Lemma 2 is false, so that, for example,

$$(30) \qquad \| x'_1, x'_s \| < \| x_1, x_s \|.$$

Since conditions (27) hold with i replaced by s, conditions (30) give the result

$$(31) \qquad \| x'_s, y' \| > \| x_s, y \|.$$

Conditions (24) and (31) imply that

$$(32) \qquad R'_i \cdot R'_s \geqq R_i \cdot R_s, \qquad (i = 2, \cdots, m+2),$$

$$(33) \qquad R \cdot R_s > R' \cdot R'_s.$$

Conditions (32) yield the result $(\lambda'_i R'_i) \cdot R'_s \geqq (\lambda'_i R_i) \cdot R_s$, ($i$ summed). Since $x'_i = x_i$, $y = y'$, we have $R'_i = R_i$, $R' = R$, whence by the previous sentence $R' \cdot R'_s \geqq R \cdot R_s$, which contradicts condition (33). Thus Lemma 2 is proved.

LEMMA 3. *Let M and M' be two sets of points, each set lying on the surface $\mathcal{K}_n$ of an $(n+1)$-dimensional Euclidean sphere E_{n+1} of radius r, and suppose that to each point $x_i \in M$ there corresponds a point $x'_i \in M'$. If the set of points M' do not all lie on the surface of a hemisphere H_{n+1} of E_{n+1}, and if conditions (24) hold for all corresponding pairs (x_i, x'_i) and (x_j, x'_j) in M and M', then conditions (25) hold for all such pairs.*

Proof. We use the same notation used in the proof of Lemma 2. Choose an *arbitrary* point in M, and without loss of generality denote it by x_1. *Suppose* that y', the point diametrically opposite to x'_1 on E_{n+1}, is such that $y' \in M'$. Let x_s be an *arbitrary* point in M. Then since

$$\| x_1, x_s \| + \| x_s, y \| = \pi r, \qquad \| x'_1, x'_s \| + \| x'_s, y' \| = \pi r,$$
$$\| x'_1, x'_s \| \leqq \| x_1, x_s \|, \qquad \| x'_s, y' \| \leqq \| x_s, y \|,$$

we must have $\| x'_1, x'_s \| = \| x_1, x_s \|$. Hence conditions (25) hold in the first case. The same proof holds if y' is a limit point of M'.

Secondly, suppose $y' \notin \bar{M}'$, the closure of M'. Let $C(M')$ be the convex hull of M'. Since $M' \notin H_{n+1}$, $C(M')$ contains the center O of E_{n+1} in its *interior*. Hence the line joining x'_1 and O meets when extended the surface of $C(M')$ at a point $P \notin \bar{M}'$. Hence let $\Delta(x'_2, \cdots, x'_{m+2})$ $(0 \leqq m \leqq n)$ be the smallest simplex of the surface of $C(M')$ containing P in its *interior*. Then the simplex $\Delta(x'_1, x'_2, \cdots, x'_{m+2})$ contains O in its interior, and hence satisfies the hypothesis of Lemma 2. Thus (25) holds, and Lemma 3 is proved. By using the closure of M', the above argument yields the

Remark. Suppose that $M' \varepsilon \mathcal{K}_n$, the surface of E_{n+1} of radius r. If each set of $m+2$ $(0 \leqq m \leqq n)$ distinct points of M' lie on a hemisphere of E_{m+1} of radius r, then M' lies on a hemisphere of E_{n+1}.

THEOREM 2. *Property* E *holds when each* $\mathfrak{M}$ *and* $\mathfrak{M}'$ *is the n-dimensional surface* $\mathcal{K}_n$ *of the* $(n+1)$*-dimensional Euclidean sphere* E_{n+1}.

The proof of this theorem falls naturally into two cases.

Case I. Suppose that the centers x'_i of the spheres $S'_i \varepsilon M'$ all lie on the surface of a closed hemisphere H_{n+1} of E_{n+1}. By Lemma 1, *each set of* $n+1$ *of the sets* $S'_i \cdot H_{n+1}$ *have a point in common.* The projection of the sets $S'_i \cdot H_{n+1}$ on the base plane of the hemisphere H_{n+1} are convex sets, which we denote by C'_i. Since a projection sets up a one-to-one correspondence between points of the surface of H_{n+1} and points of the base plane, each set of $n+1$ of the sets C'_i have a point in common. *By a theorem of Helly*[4] [2], *or* [1, p. 297], *it follows that all the sets* C'_i *have a point in common.* Thus on account of the one-to-one correspondence, $H_{n+1} \cdot \Pi_{M'} S'_i \neq 0$, and Theorem 2 is proved for this case.

Case II. In this case suppose that the centers x'_i do *not* all lie on the surface of a hemisphere of E_{n+1}. Then by Lemma 3, $\| x'_i, x'_j \| = \| x_i, x_j \|$. Hence by a rigid motion of M' in $\mathcal{R}_{n+2}$ we can make $x'_i = x_i$ for all corresponding pairs S_i and S'_i. Since $r_i = r'_i$, we have $S_i = S'_i$. Thus $\Pi_M S_i \neq 0$ is identical with $\Pi_{M'} S'_i \neq 0$. Consequently Theorem 2 has been proved in all cases.

4. Property E **in** $\mathfrak{H}$**.** Let $\mathfrak{H}$ be a general Hilbert space, the number of dimensions not necessarily being denumerable.

THEOREM 3. *Property* E *holds when each* $\mathfrak{M}$ *and* $\mathfrak{M}'$ *is a Hilbert space* $\mathfrak{H}$.

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[4] *Loc. cit.*[2]

This theorem is a consequence of the following statement. *If each finite set of the spheres S_i in $\mathfrak{H}$ have a point in common, then they all have a point in common.* To prove this let the spheres S_i be well-ordered, and let the corresponding products $\prod_{i=1}^{\alpha} S_i = T_\alpha$, ($\alpha$ = transfinite ordinal). Since the sets T_α are well-ordered, *suppose that the first* one which is zero is T_β. If β is a limiting ordinal, then since the space $\mathfrak{H}$ is regular, and since the sets T_α are a transfinite bounded monotone decreasing sequence of closed convex sets, a theorem of Smulian [4] states that $T_\beta \neq 0$, contrary to our assumption.[5] The remaining case is that in which the first of the T_α which is zero is T_β, with $\beta = \delta + n$, where δ is a limiting ordinal and n is an integer. In this case reorder the spheres S_i, so that the first set of the sets $\prod_{i=1}^{\alpha} S_i = T_\alpha{}^1$ which is zero is $T_{\beta_1}{}^1$, where $\beta_1 \leqq \delta$. If β_1 is *not* a limiting ordinal, repeat this process so as to obtain a sequence of sets $T_{\beta_\lambda}{}^\lambda$ (β_λ and λ are ordinals), so that if $\lambda > \mu$, $\beta_\lambda < \beta_\mu$. The ordinals β_λ must then have a smallest member, denoted by β_γ. The number β_γ must, by construction, be either a limiting ordinal or an integer. In the former case we have, by the Smulian theorem [4], $T_{\beta_\gamma}{}^\gamma \neq 0$, whereas in the latter case this condition follows by assumption. Since in all cases $T_\alpha{}^\lambda \neq 0$, ($\lambda$ = fixed ordinal, α = any ordinal), the statement above is proved.

In a Hilbert space any $n + 1$ spheres S_i must have intersections in the same n-dimensional Euclidean space $\mathfrak{R}_n$. Since for any $n + 1$ spheres S_i in $\mathfrak{H}$, the n-dimensional Euclidean spheres $S_i \cdot \mathfrak{R}_n$, $S'_i \cdot \mathfrak{R}'_n$ satisfy the hypotheses of Theorem 1, we must have $\prod_{i=1}^{n+1} S'_i \neq 0$. Since n is an arbitrary integer, by the italicized theorem in the preceding paragraph we must have $\Pi_{M'} S'_i \neq 0$. Hence Theorem 3 is proved.

5. The extension.

THEOREM 4. *Let $f(x)$ be a function which maps a set S in a metric space $\mathfrak{M}$ into a set S' in a metric space $\mathfrak{M}'$, so as to preserve the Lipschitz condition. Furthermore suppose that $\mathfrak{M}$ and $\mathfrak{M}'$ have the property* E. *Then $f(x)$ can be extended to any set $T \supset S$ so as to preserve the Lipschitz condition.*

A Lipschitz extension is possible as soon as one establishes the extensibility of $f(x)$ from an *arbitrary set U to any additional point x_0*. To do this

[5] The Smulian theorem and its applicability here were suggested to me by my colleague Max Zorn. The italicized statement in the above proof also holds when $\mathfrak{H}$ is a regular Banach space and when S_i are bounded, closed, convex sets.

here let S_i be spheres in $\mathfrak{M}$ with centers x_i and radii $r_i = \| x_0, x_i \|$, respectively, and let S'_i be spheres in $\mathfrak{M}'$ with centers $x'_i \equiv f(x_i)$ and radii $r'_i = r_i$ respectively. Let x_i range over U and denote the corresponding set of spheres by M. Since we have assumed that property E holds, any point $x'_0 \varepsilon \Pi_{M'} S'_i \neq 0$ serves as an extension $f(x_0)$, for $\| x'_0, f(x_i) \|' \leqq \| x_0, x_i |$ for all corresponding pairs $[x_i, f(x_i)]$ in U and U'. If the spaces $\mathfrak{M}$ and $\mathfrak{M}'$ are separable, the function $f(x)$ can be extended to a denumerable dense set by ordinary induction, and then to the whole space by passage to the limit. For example, see [5, p. 106]. If $\mathfrak{M}$ and $\mathfrak{M}'$ are not separable, the extension follows by transfinite induction.

THEOREM 5. *The extension described in Theorem 4 holds for the cases;* (1) $\mathfrak{M} = \mathfrak{M}' = \mathcal{R}_n$; (2) $\mathfrak{M} = \mathfrak{M}' = \mathcal{K}_n$; (3) $\mathfrak{M} = \mathfrak{M}' = \mathcal{H}$. *Furthermore for cases* (1) *and* (3) *the extension* $[f(x), x \varepsilon T]$ *can be defined so as to be contained in any closed, convex set* $L \supset S'$.

The first statement follows from Theorem 4, and the fact that $\mathcal{R}_n$, $\mathcal{K}_n$ and $\mathcal{H}$ have property E.

To prove the concluding statement for the space $\mathcal{R}_n$, let S'_i $(i = 1, \cdots, n+1)$ be any $n+1$ spheres in M'. Then the simplex $\Delta(x'_1, \cdots, x'_{n+1}) \subset L$. Since in the proof of Lemma 1, $\Delta(x'_1, \cdots, x'_{n+1}) \cdot \prod_{i=1}^{n+1} S'_i \neq 0$, we have $L \cdot \prod_{i=1}^{n+1} S'_i \neq 0$. This together with the fact that $\Pi_{M'} S'_i \neq 0$, implies by a theorem of Helly [6] [2], that $L \cdot \Pi_{M'} S'_i \neq 0$. Thus we can choose $x'_0 \varepsilon L \cdot \Pi_{M'} S'_i$, so that the extension $f(x_0) \varepsilon L$. By induction, it follows that the total extension can be defined so that $[f(x), x \varepsilon T] \subset L$.

A corresponding result holds for a Hilbert space $\mathcal{H}$, since by the preceding reasoning we have $L \cdot \prod_{i=1}^{n+1} S'_i \neq 0$, where n is an *arbitrary* integer. Hence by virtue of the italicized statement in the proof of Theorem 3, we have $L \cdot \Pi_{M'} S'_i \neq 0$, and the rest of the argument is the same as that for $\mathcal{R}_n$.

COROLLARY. *Suppose that the function* $f(z)$ *maps a set* S *in* $\mathcal{R}_n$ *into a set* S' *in* $\mathcal{R}_n$, *and suppose that there exists a constant* $K > 0$ *so that*

$$\| f(z_1), f(z_2) \| \leqq K \| z_1, z_2 \| \tag{34}$$

for all pairs z_1 *and* z_2 *in* S. *Then* $f(z)$ *can be extended to any set* $T \supseteq S$ *so as to preserve condition* (34). *This also holds for a Hilbert space* $\mathcal{H}$. *The concluding statement of Theorem 5 also holds.*

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[6] *Loc. cit.*[2]

This is an immediate consequence of Theorem 4 and of the fact that $k \| z_1, z_2 \| = \| kz_1, kz_2 \|$. In the proof of Theorem 4, let $x_i \equiv kz_i$, $x'_i \equiv f(z_i)$, $x_0 \equiv kz_0$, $r_i = r'_i \equiv \| x_0, x_i \|$. Then by the argument in the proof of Theorem 4, the extension $f(z_0)$ exists. The rest of the proof is the same as that given for Theorems 4 and 5.

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NÖRLUND METHODS OF SUMMABILITY THAT INCLUDE THE CESÀRO METHODS OF ALL POSITIVE ORDERS.*

By J. D. Hill.

1. Introduction. In a recent paper [1] H. L. Garabedian has established the existence of Hausdorff methods of summability having the unusual property of including the Cesàro methods of all real and positive orders. The object of this paper is to consider the corresponding problem for Nörlund methods. We begin by recalling a few facts on which the sequel depends.

The Nörlund method of summability $[N;\{p_k\}]$ is defined by a triangular matrix (a_{mk}) whose elements are of the form $a_{mk}\equiv p_{m-k}/P_m$ $(k=0,1,2,\cdots,m;$ $m=0,1,2,\cdots)$, where $\{p_k\}$ is a sequence of complex numbers such that $P_m\equiv p_0+p_1+\cdots+p_m$ is different from zero for all m. One generally takes $p_0=1$ since no loss of generality results. We then have $a_{mm}=P_m^{-1}\neq 0$ for all m, so that the Nörlund transformation

$$t_m=P_m^{-1}\sum_{k=0}^{m}p_{m-k}s_k \qquad (m=0,1,2,\cdots), \tag{1.1}$$

is *reversible*; [2] that is to say, corresponding to each convergent sequence $\{t_m\}$ there exists a unique (convergent or divergent) sequence $\{s_k\}$ satisfying the equations (1.1).

The Silverman-Toeplitz regularity conditions for $[N;\{p_k\}]$ reduce to

$$\sum_{k=0}^{m}|p_k|=O(P_m) \quad \text{and} \quad p_m=o(P_m) \quad \text{as} \quad m\to\infty. \tag{1.2}$$

Our results are based on the following general theorem of S. Mazur.[3]

Theorem of Mazur. *Let A with matrix (a_{mk}) be a given reversible*

* Received January 17, 1944.

[1] H. L. Garabedian, "Hausdorff methods of summation which include all of the Cesàro methods," *Bulletin of the American Mathematical Society*, vol. 48 (1942), pp. 124-127.

[2] S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 90.

[3] S. Mazur, "Über lineare Limitierungsverfahren," *Mathematische Zeitschrift*, vol. 28 (1928), pp. 599-611. For the theorem in the form stated here in which Mazur's hypothesis of *normality* is replaced by that of *reversibility*, see J. D. Hill, "Some properties of summability," *Duke Mathematical Journal*, vol. 9 (1942), pp. 373-381; in particular, p. 376.

method, and let B with matrix (b_{mk}) be any method whatsoever. Let the sequences $\{\xi_k\}$ and $\{\xi_k{}^n\}$ be defined as the (unique) solutions of the equations

$$(1.3)\qquad \sum_{k=0}^{\infty} a_{mk}\xi_k = 1 \qquad (m = 0, 1, 2, \cdots);$$

$$(1.4)\qquad \sum_{k=0}^{\infty} a_{mk}\xi_k{}^n = \delta_m{}^n \qquad (m, n = 0, 1, 2, \cdots);$$

where $\delta_m{}^n$ is the Kronecker symbol. Then in order that B include A the following conditions are necessary and sufficient.

$$(1.5)\qquad B - \lim \{\xi_k\} \text{ exists and } = 1;$$

$$(1.6)\qquad B - \lim \{\xi_k{}^n\} \text{ exists and } = 0 \qquad (n = 0, 1, 2, \cdots);$$

$$(1.7)\qquad \sup_r \{Q_r{}^m \equiv \sum_{n=0}^{\infty} |\sum_{k=0}^{r} b_{mk}\xi_k{}^n|\} < \infty \qquad (m = 0, 1, 2, \cdots);$$

$$(1.8)\qquad \sup_m \{Q_m \equiv \sum_{n=0}^{\infty} |\sum_{k=0}^{\infty} b_{mk}\xi_k{}^n|\} < \infty.$$

2. The condition that $[N; \{p_k\}] \supset (C; \alpha > 0)$. We shall now apply the theorem of Mazur with $A \equiv (C; \alpha > 0)$ and with $B \equiv [N; \{p_k\}]$, a regular Nörlund method. Since $(C; \alpha)$ is itself a (reversible) regular Nörlund method $[N; \{p_k(\alpha)\}]$ where

$$p_k(\alpha) \equiv \binom{k + \alpha - 1}{\alpha - 1} \text{ and } P_k(\alpha) = \binom{k + \alpha}{\alpha} \qquad (k = 0, 1, 2, \cdots),$$

we see that (a_{mk}) in the preceding theorem becomes a triangular matrix whose elements are $p_{m-k}(\alpha)/P_m(\alpha)$; and that (b_{mk}) is to be replaced by (p_{m-k}/P_m). With these replacements it follows immediately from (1.3) that $\xi_k \equiv \xi_k(\alpha) = 1$ $(k = 0, 1, 2, \cdots)$, and in (1.4) it is known that [4]

$$\xi_k{}^n \equiv \xi_k{}^n(\alpha) = (-1)^{k-n}\binom{n + \alpha}{\alpha}\binom{\alpha}{k - n} \qquad (k, n = 0, 1, 2, \cdots),$$

where the final binomial coefficient is defined as 0 if $k - n < 0$. Since $\lim_k \xi_k(\alpha) = 1$ and $\lim_k \xi_k{}^n(\alpha) = 0$ $(n = 0, 1, 2, \cdots)$ we observe that (1.5) and (1.6) are automatically satisfied on account of the assumed regularity of $[N; \{p_k\}]$. Condition (1.7) is likewise automatically fulfilled since for each m, $Q_r{}^m(\alpha)$ is constant for fixed α and $r > m$.

Our problem is consequently reduced to a study of the conditions under which (1.8) will be satisfied. We find in the present case that

[4] J. D. Hill, "On perfect methods of summability," *Duke Mathematical Journal*, vol. 3 (1937), pp. 702-714; in particular, pp. 705-707.

$$(2.1)\qquad Q_m(\alpha) = |P_m|^{-1} \sum_{n=0}^{m} \binom{n+\alpha}{\alpha} \left| \sum_{j=0}^{m-n} (-1)^j \binom{\alpha}{j} p_{m-n-j} \right|.$$

To assist in the analysis of this expression we introduce the following power series:

$$(2.2)\qquad p(z) \equiv \sum_{k=0}^{\infty} p_k z^k;$$

$$(2.3)\qquad C_\alpha(z) \equiv (1-z)^\alpha p(z) \equiv \sum_{n=0}^{\infty} c_n(\alpha) z^n,$$

so that

$$(2.4)\qquad c_n(\alpha) = \sum_{j=0}^{n} (-1)^j \binom{\alpha}{j} p_{n-j};$$

$$(2.5)\qquad C^*_\alpha(z) \equiv \sum_{n=0}^{\infty} |c_n(\alpha)| z^n;$$

$$(2.6)\qquad (1-z)^{-\alpha-1} C^*_\alpha(z) \equiv \sum_{m=0}^{\infty} g_m(\alpha) z^m,$$

so that

$$(2.7)\qquad g_m(\alpha) = \sum_{n=0}^{m} \binom{n+\alpha}{\alpha} |c_{m-n}(\alpha)|.$$

Since the series in (2.2) has a positive radius of convergence [5] it is evident that the same is true of the series in (2.3), (2.5), and (2.6). In view of the foregoing definitions we observe that (2.1) reduces to the form $Q_m(\alpha) = |P_m|^{-1} g_m(\alpha)$. The condition (1.8) is therefore equivalent to the condition

$$(2.8)\qquad g_m(\alpha) = O(P_m) \qquad (m \to \infty).$$

We express the results of the preceding discussion in the form of the following theorem.

THEOREM 1. *In order that a given regular* $[N; \{p_k\}]$ *include* $(C; \alpha)$ *for a given* $\alpha > 0$ *it is necessary and sufficient that the condition* (2.8) *be satisfied.*

COROLLARY. *If a regular* $[N; \{p_k\}]$ *is independent of* α *and if* (2.8) *holds for each* $\alpha > 0$, *then* $[N; \{p_k\}]$ *includes* $(C; \alpha)$ *for all positive* α.

To illustrate the applicability of Theorem 1 the reader may easily verify the well known result

$$(2.9)\qquad (C; \beta) \supset (C; \alpha) \qquad (\beta > \alpha > 0).$$

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[5] See footnote 4, p. 706.

In the next section we derive Theorem 2 as a special case of the preceding corollary, and use it to give an example of a method $[N;\{p_k\}]$ that includes all of the $(C;\alpha>0)$.

3. An example of $[N;\{p_k\}]$ that includes $(C;\alpha)$ for all $\alpha>0$. If for a fixed $\alpha>0$ the condition

$$c_n(\alpha)\geqq 0 \qquad (n=0,1,2,\cdots), \tag{3.1}$$

holds for a given regular Nörlund method, it follows at once from (2.3), (2.5), and (2.6) that

$$\sum_{m=0}^{\infty} g_m(\alpha)z^m=(1-z)^{-1}p(z)=\sum_{m=0}^{\infty}P_m z^m.$$

Consequently, $g_m(\alpha)=P_m$ $(m=0,1,2,\cdots)$, so that (2.8) in this event is trivially satisfied. On the other hand, for $n=1$ the relations (2.4) and (3.1) yield $c_1(\alpha)=p_1-\alpha\geqq 0$ or $p_1\geqq\alpha$. Thus if $[N;\{p_k\}]$ is independent of α condition (3.1) can not hold for all values of α. In view of these circumstances we are led to seek some modification of (3.1) and the following one proves to be useful. We shall say that a Nörlund method satisfies *Condition S* if, and only if, (i) it is regular; (ii) independent of α; and (iii) there exists an integer $r=r(\alpha)$ such that the condition

$$c_n(\alpha)\geqq 0 \text{ for all } n\geqq r(\alpha), \tag{3.2}$$

is satisfied for each $\alpha>0$.

We now state the following theorem.

THEOREM 2. *If $[N;\{p_k\}]$ satisfies Condition S and if*

$$m^a=O(P_m) \qquad (m\to\infty), \tag{3.3}$$

for each $\alpha>0$, then $[N;\{p_k\}]$ includes $(C;\alpha)$ for all positive α.

Proof. From (3.2), (2.3), and (2.5) it follows that

$$C^*_a(z)=\sum_{n=0}^{r-1}h_n(\alpha)z^n+C_a(z),$$

where $h_n(\alpha)=|c_n(\alpha)|-c_n(\alpha)$ for $(n=0,1,2,\cdots,r-1)$. Then (2.6) becomes

$$\sum_{m=0}^{\infty}g_m(\alpha)z^m=\sum_{m=0}^{\infty}\left\{\sum_{j=0}^{m}h_j(\alpha)\binom{m-j+\alpha}{\alpha}\right\}z^m+P(z),$$

where $P(z)\equiv(1-z)^{-1}p(z)=\Sigma P_m z^m$. Hence for $m\geqq r-1$ we obtain

$$|g_m(\alpha) - P_m| = \left|\sum_{j=0}^{r-1} h_j(\alpha)\binom{m-j+\alpha}{\alpha}\right| \leqq \binom{m+\alpha}{\alpha}\sum_{j=0}^{r-1}|h_j(\alpha)|.$$

Therefore

$$(3.4) \qquad g_m(\alpha) \leqq H_\alpha\binom{m+\alpha}{\alpha} + |P_m| \qquad (m \geqq r-1),$$

where $H_\alpha \equiv \sum_{j=0}^{r-1}|h_j(\alpha)|$. Finally, by means of (3.4) and the well known relation

$$\binom{m+\alpha}{\alpha} \cong \frac{m^\alpha}{\Gamma(\alpha+1)} \qquad (\alpha \neq -1, -2, -3, \cdots; m \to \infty),$$

we see that (3.3) implies (2.8). The theorem then follows from the corollary to Theorem 1.

Example. The Nörlund method $[N; \{\cosh k^{\frac{1}{2}}\}]$ *satisfies the conditions of Theorem* 2. For since $p_k \equiv \cosh k^{\frac{1}{2}} > 0$ $(k = 0, 1, 2, \cdots)$ the regularity conditions (1.2) reduce to $p_m = o(P_m)$. But this condition is easily seen to be satisfied since the inequality

$$P_m \geqq 1 + \int_0^m \cosh x^{\frac{1}{2}}\,dx = 2m^{\frac{1}{2}}\sinh m^{\frac{1}{2}} - 2\cosh m^{\frac{1}{2}} + 3,$$

immediately yields the estimate $p_m/P_m = O(m^{-\frac{1}{2}})$. Part (i) of Condition S is therefore fulfilled.

Since part (ii) is obvious we pass on to part (iii). On account of (2.9) we may clearly restrict α to integral values. We take $r(\alpha) \equiv \alpha$ and set $\phi(x) \equiv \cosh x^{\frac{1}{2}}$ so that $p_k = \phi(k)$ for $(k = 0, 1, 2, \cdots)$. Then in the usual difference notation (2.4) becomes $c_n(\alpha) = \Delta^\alpha\phi(n-\alpha)$ for $n \geqq \alpha$. By a familiar elementary theorem $\Delta^\alpha\phi(n-\alpha) = \phi^{(\alpha)}(n-\alpha\theta)$ where $0 < \theta < 1$. Since the Maclaurin expansion of $\phi(x)$ shows that all of its derivatives are positive for $x > 0$, we conclude that (3.2) is satisfied.

Finally, to see that (3.3) holds we have merely to observe that $P_m^{-1}m^\alpha < p_m^{-1}m^\alpha < 2m^\alpha e^{-m^{\frac{1}{2}}} = o(1)$ for each α as $m \to \infty$.

Consequently, $[N; \{\cosh k^{\frac{1}{2}}\}]$ is an example of a Nörlund method that includes $(C; \alpha)$ for all positive α.

As a final remark it is of interest to notice that Abel summability (A), on the other hand, includes all regular Nörlund methods corresponding to real sequences[6] $\{p_k\}$. Thus we have the relation $(C; \alpha) \subset [N; \{\cosh k^{\frac{1}{2}}\}] \subset (A)$ for all $\alpha > 0$.

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[6] G. F. Woronoi–J. D. Tamarkin, "Extensions of the notion of the limit of the sum of terms of an infinite series," *Annals of Mathematics*, Second series, vol. 33 (1932), pp. 422-428; in particular, p. 426.

MONOTONE DECOMPOSITIONS OF CONTINUA.*

By M. E. Shanks.

A continuous transformation of a T_1 space (points are closed sets [6, p. 59]) generates a decomposition of the space into disjoint closed sets which fill up the space. Conversely, every such decomposition defines one or more hyperspaces, whose points are the sets of the decomposition, depending on the manner in which these sets are topologized. For each such hyperspace there is a continuous transformation of the given space which takes a point into the set to which it belongs. A discussion of different types of decompositions and their hyperspaces can be found in [6].

Until recently, there had been no investigation of the class of all such decompositions without regard for the different possible topologies that might be put on the hyperspace. In a recent paper B. H. Arnold [1] considered the class of all upper semi-continuous (u. s. c.) decompositions [6, p. 67; 4, p. 123] of a T_1 space and showed that it is possible to construct a space from the decompositions, which are partially ordered, homeomorphic to the given space. In another paper [2] the author obtained results on the semi-linear space of all metrics on a compactum, compatible with its topology, by considering the lattice of all u. s. c. decompositions of the space.

In the same order of ideas, it is natural to consider special types of decompositions and to inquire as to how their structure as an ordered system reflects the topological structure of the space. In this paper there will be considered monotone decompositions of compacta, and especially continua, for the study of whose structure they are particularly well suited.

It is shown that two continua are homeomorphic if and only if their lattices of monotone u. s. c. decompositions are isomorphic under a restricted type of isomorphism. The restriction is essential as it is shown that the lattice of monotone decompositions on a dendrite, or a linear graph, is isomorphic with that for an arc.

A wide class of continua, which we call generalized dendrites, is defined and characterized as those continua for which the monotone u. s. c. decom-

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positions form a sublattice of the lattice of all u. s. c. decompositions. This class includes dendrites and Knaster continua, and both types make the lattice of monotones distributive. Finally, a characterization of dendrites is obtained. The paper concludes with some examples and problems.

1. The lattice of monotone decompositions. Denote by $\mathfrak{D}_1(X)$ the class of all decompositions of the compactum X into disjoint closed sets. The sets of a decomposition are called *slices*. Partially order $\mathfrak{D}_1(X)$ as follows: if $D_1, D_2 \varepsilon \mathfrak{D}_1(X)$, then $D_1 < D_2$ if each slice of D_2 is *contained in* some slice of D_1. Then, as is easy to show, $\mathfrak{D}_1(X)$ is a complete lattice. Denote by $\mathfrak{D}(X)$ the set of all u. s. c. decompositions of X with the same ordering as before. It is shown in [2] that $\mathfrak{D}(X)$ is also a complete lattice. The join, $D_1 \vee D_2$, of two decompositions is the decomposition whose slices are the intersections of slices of D_1 and D_2. Their meet, $D_1 \wedge D_2$, is the join of all decompositions less than both.

The ordering chosen here is opposite to that used by Arnold and may seem unnatural. Nevertheless, it is natural in the sense that the natural ordering of the quasi-metrics on X induces the above ordering of the decompositions [2].

Denote by $\mathfrak{D}_m(X)$ the subset of $\mathfrak{D}(X)$ consisting of all monotone decompositions, i. e. those whose slices are continua, degenerate or not. Then $\mathfrak{D}_m(X)$ will not in general be a sublattice of $\mathfrak{D}(X)$ since in general the intersection of continua is not a continuum. However $\mathfrak{D}_m(X)$ will be a lattice with respect to the order relation in it. The details are left to the reader.[1]

Since $\mathfrak{D}(X)$ is complete, it has a least element O which has a single slice equal to X, and a greatest element I whose slices are all points. I is always monotone while O is monotone only if X is a continuum. Slices consisting of but a single point are *degenerate*. A decomposition is said to be *simple* if it has but one non-degenerate slice. Let $\mathfrak{D}_{sm}(X)$ denote the class of all simple monotone decompositions, which are necessarily upper semi-continuous.

THEOREM 1. 1. *The continua X and Y are homeomorphic*[2] *if and only*

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[1] That it is actually a lattice will not be needed in this paper. The join in $\mathfrak{D}_m(X)$ of two monotone decompositions is merely the monotone factor of the monotone-light factorization of their join in $\mathfrak{D}(X)$. In other words, the slice containing any point x is the component containing x of the intersection of the slices of the given decompositions which contain x.

[2] It would be desirable to give lattice conditions on $\mathfrak{D}_m(X)$ which would insure that X be a continuum and then reconstruct the space from the lattice. This is one of a class of unsolved problems of characterizing properties of a space in terms of the decomposition lattice or the monotone decomposition lattice.

if there is an isomorphism of $\mathfrak{D}_m(X)$ *onto* $\mathfrak{D}_m(Y)$ *which carries* $\mathfrak{D}_{sm}(X)$ *onto* $\mathfrak{D}_{sm}(Y)$.

Proof. To each point x of X associate the class ξ of all elements of $\mathfrak{D}_{sm}(X)$ which contain x in their non-degenerate slice. This class, ξ, is maximal with respect to the property that the meet of any finite number of its elements is in $\mathfrak{D}_{sm}(X)$, and in fact in ξ. This characterizes ξ since the slices of any such maximal collection form a maximal collection of continua with the finite intersection property, and hence a common point. Under the isomorphism τ, ξ transforms into a collection η of elements of $\mathfrak{D}_{sm}(Y)$ which is also maximal with respect to the above property, and thus determines a point y of Y which is clearly unique.[3] There is thus defined a transformation T of X into Y. Because τ is an isomorphism it is easy to see that T is one-one and onto Y.

To establish the continuity of T we note that if C is a continuum in X and $C' = T(C)$, then C' is also a continuum. For, C determines a simple monotone decomposition D which belongs to each ξ associated with the points x of C, and similarly for $D' = \tau(D)$ as regards C'. Thus C' is a continuum.

Suppose $x_1, \cdots, x_n, \cdots$, is a sequence of points of X approaching x, and $\lim y_n = y \neq T(x)$, where $y_n = T(x_n)$. If T is not continuous at x this situation prevails for some sequence of points x_n.

Case 1. If X is locally connected at x, or infinitely many x_n are on a locally connected subcontinuum containing x, there is a monotonically decreasing sequence of continua, C_n, closing down on x, such that each C_n contains infinitely many x_n. Thus $C'_n = T(C_n)$ contains infinitely many y_n and hence y, so each C'_n contains y and $T(x)$. But to C_n and C'_n there correspond simple monotone decompositions D_n and D'_n respectively. Since the C_n close down on x the D_n have a join equal to I, while the D'_n will have a join which is a simple monotone decomposition whose non-degenerate slice is the intersection of all the C'_n, and unless $y = T(x)$ this join would not be I.

Case 2. Suppose X is not locally connected at x and no locally connected subcontinuum contains infinitely many x_n. There exist then non-degenerate

[3] It must be noted that ξ contains simple monotone decompositions with arbitrarily small slices which are non-degenerate. This insures that the finite intersection property holds. The fact that the slices can be arbitrarily small is reflected in the lattice by the property that ξ contains a sequence increasing, in the sense of the order, to I. Thus η contains such a sequence.

pairwise disjoint continua, C_n, $(n = 1, 2, \cdots)$, containing[4] x_n and converging to a continuum C containing x, $\lim C_n = C$. Consider $C'_n = T(C_n)$ and suppose that the continua C'_n converge to a continuum[4] C'. If δ is the greatest diameter of C, C_n, $(n = 1, 2, \cdots)$, then as $\delta \to 0$, $T(C)$, C'_n, and C' must also have their diameters approach zero, for otherwise the associated decompositions would not approach I. So, for δ sufficiently small $T(C)$ is distinct from C'. Suppose that this is the case.

Let D_0 be the monotone decomposition whose non-degenerate slices are C_n, $(n = 1, 2, \cdots)$, and C. Then $D'_0 = \tau(D_0)$ has the non-degenerate slices C'_n, $(n = 1, 2, \cdots)$, C', and $T(C)$. Since $T(C)$ is disjoint from C', one gets another u. s. c. decomposition D_0'' greater than D_0' by allowing all points of $T(C)$ to be point slices. Therefore $\tau^{-1}(D_0'')$ is greater than D_0 and has for its non-degenerate slices C_n, $(n = 1, 2, \cdots)$, but not C. But this decomposition fails to be u. s. c. This completes the proof that the specified isomorphism implies the homeomorphism. The converse is obvious.

An examination of the above proof for the case of locally connected continua gives the following theorem.

THEOREM 1.2. *The locally connected continua X and Y are homeomorphic if and only if $\mathfrak{D}_{sm}(X)$ is isomorphic to $\mathfrak{D}_{sm}(Y)$.*

If X is not a continuum Theorem 1.1 is false, for $\mathfrak{D}_m(X)$ reduces to a single element if X is 0-dimensional. Also, the assumption that the simple monotone decompositions correspond cannot be relaxed as is shown by the results of the next section.

2. Factorization theorems.

THEOREM 2.1. *If $X = X_1 + X_2$, where X, X_1, X_2, are compacta, and X_1 and X_2 have at most a finite[5] set of points in common, then $\mathfrak{D}_m(X)$ is isomorphic to the direct product of $\mathfrak{D}_m(X_1)$ and $\mathfrak{D}_m(X_2)$.*

Proof. If D is any monotone u. s. c. decomposition of X and S any slice of D, then $S = S_1 + S_2$, where $S_i = S \cdot X_i$, $i = 1, 2$. Let D_i, $(i = 1, 2)$, be the monotone decomposition of X_i whose slices are the components of S_i for all S. To see that D_i is u. s. c., suppose that $\lim\inf S_i^n \cdot S_i \neq 0$, as $n \to \infty$,

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[4] By possibly choosing a subsequence.

[5] The following example shows that $X_1 \cdot X_2$ may not be uncountable. Let X_1 and X_2 be two arcs intersecting in a perfect set non-dense on either arc. It can be shown that the lattice of monotones on X is not modular, while the direct product of the monotones on X_1 and X_2 is distributive. The countable case is open.

where S_i^n, S_i are the slices of D_i contained in the slices S_i, S of D respectively. Then $\limsup S_i^n \subset S = S_1 + S_2$. But $\limsup S_i^n \subset X_i$ so that $\limsup S_i^n \subset S_i$.

We have thus a *natural* mapping, τ, of $\mathfrak{D}_m(X)$ into $\mathfrak{D}_m(X_1) \times \mathfrak{D}_m(X_2)$. To see that τ is a mapping "onto" the direct product, suppose $D_i \varepsilon \mathfrak{D}_m(X_i)$, $(i = 1, 2)$, and $x_1, \cdots, x_n$ are the points common to X_1 and X_2. If X_1 and X_2 have no points in common the proof is immediate. Each $x_i \varepsilon S_i', S_i''$, where S_i', S_i'' are slices of D_1 and D_2 respectively. The set $F = \sum_{n-1} (S_i' + S_i'')$ is the sum of a finite number of continua. Now define a decomposition D of X whose slices are the same as those of D_1 and D_2 together except for S_i', S_i'', $(i = 1, \cdots, n)$. These latter slices are replaced by the components of F. D is obviously monotone and is easily shown to be upper semi-continuous. It is apparent that τ maps D on the product of D_1 and D_2. It is also easy to see that τ is one-one since the intersection of X_1 and X_2 is finite.

COROLLARY. *If A is an arc, then $\mathfrak{D}_m(A)$ is isomorphic to $\mathfrak{D}_m(A) \times \mathfrak{D}_m(A)$.*

THEOREM 2.2. *If A is an arc, then the direct product of countably many $\mathfrak{D}_m(A)$ is isomorphic to $\mathfrak{D}_m(A)$.*

Proof. For simplicity suppose that A is the unit interval $(0, 1)$. It will suffice to decompose A into a countable set of arcs, $\{A_n\}$, such that each monotone decomposition defines a unique decomposition of each A_n, and conversely. Such a set is the following: $A_1 = (0, 1/2)$, $A_2 = (1/2, 3/4), \cdots, A_n = ((2^n - 1)/2^n, (2^{n+1} - 1)/2^{n+1}), \cdots$.

THEOREM 2.3. *If $\mathfrak{D}_m(X)$ is isomorphic to $\mathfrak{D}_m(A)$, where A is an arc and X is a continuum, then X is hereditarily locally connected.*[6]

Proof. Let τ be such an isomorphism, $\tau(\mathfrak{D}_m(X)) = \mathfrak{D}_m(A)$. If X is not hereditarily locally connected it contains a continuum of convergence C_0. which is the limit of disjoint continua C_n, $(n = 1, 2, \cdots)$. Denote by D the monotone u. s. c. decomposition whose only non-degenerate slices are the C_n, and by D_n the simple monotone decomposition whose non-degenerate slice is C_n.

Now the join of D_i and D_j, if $i \neq j$, is I and so also for $\tau(D_i)$ and $\tau(D_j)$. The non-degenerate slices of $\tau(D_i)$ and $\tau(D_j)$ therefore do not overlap. Consider the decomposition, D^*, of A which is the meet of $\tau(D_i)$, $(i = 1, 2, \cdots)$.

[6] The converse is false for the continuum of the previous footnote is hereditarily locally connected, and in fact regular.

Since the non-degenerate slices of $\tau(D_i)$, $i \geqq 1$, and $\tau(D_0)$ do not overlap, the same is true for D^* and $\tau(D_0)$. Thus $\tau(D) < D^* < \tau(D_i)$ if $i \geqq 1$. Now D^* is u. s. c. since it is a monotone decomposition of an arc, and so $\tau^{-1}(D^*)$ is also. Thus $D < \tau^{-1}(D^*) < D_i$, if $i \geqq 1$, and the non-degenerate slices of $\tau^{-1}(D^*)$ are the same as those of D_i, $i \geqq 1$, but different if $i = 0$. But then C_0 is not a full slice of $\tau^{-1}(D^*)$ and the latter is not upper semi-continuous. This contradiction completes the proof.

An example exists of a continuum which is hereditarily locally connected, and in fact regular, for which $\mathfrak{D}_m(X)$ is not isomorphic to $\mathfrak{D}_m(A)$.

THEOREM 2.4. *If X is a linear graph or a dendrite then $\mathfrak{D}_m(X)$ is isomorphic to $\mathfrak{D}_m(A)$, where A is an arc.*

Proof. If X is a linear graph, it is the sum of a finite number of arcs with a finite set of points in common, and one may apply Theorem 2.1 and its corollary.

If X is a dendrite, it is the sum of a countable set of arcs p_iq_i, $(i = 1, 2, \cdots)$, and a set E of end points, such that p_iq_i is a null sequence, and $p_nq_n \cdot \sum_1^{n-1} p_iq_i = q_n$ [4, p. 89]. Denote by X_n the connected linear graph $\sum_1^n p_iq_i$.

If D_i is a monotone decomposition of p_iq_i, for $(i = 1, 2, \cdots)$, we show that these decompositions determine a decomposition D of X which coincides with D_i on p_iq_i. The decompositions D_i, $(i = 1, \cdots, n)$, determine a unique monotone decomposition D^n of X_n since X_n is a linear graph. If $x \in X_n$ and is contained in the slice S_n of D^n, the addition of more arcs p_iq_i can only increase the slice containing x. Let $S = \overline{\sum_{i=n, n+1, \ldots}^{\infty} S_i}$, and consider all the sets, or "slices" S. These fill up X except possibly for some end points. If an end point is not in some S, then define the slice containing it to be degenerate. The sets S are continua and furthermore are disjoint. For, if S_n, S'_n are disjoint slices of D^n, then S and S' cannot have a point in common. If they were to, this point would not be in X_n and there would exist an arc in X exterior to X_n joining S_n and S'_n, but this would give a simple closed curve in X. Therefore the sets S, regarded as slices, define a monotone decomposition D. It is clear that D is u. s. c., since the slices form a null family. D coincides with D_i on p_iq_i by construction.

Conversely, if D is a monotone decomposition of X, then D defines on p_iq_i a unique decomposition D_i. The slices of D_i which intersect X_n define slices of D^n which are unique. These decompositions D^n determine D as

described above. To see this, suppose that the D^n were to determine a decomposition D', different from D. Then D' and D would differ on some arc of X and therefore on one of the arcs p_iq_i. This is a contradiction.

3. Generalized dendrites. Up to this point we have not considered the structure of $\mathfrak{D}_m(X)$ in relation to the containing lattice, $\mathfrak{D}(X)$, of all u. s. c. decompositions. We observe that there are two distinct problems; the structure of $\mathfrak{D}_m(X)$ by itself as a partially ordered system, and its structure as a subsystem of $\mathfrak{D}(X)$. In addition, one may embed both in $\mathfrak{D}_1(X)$, the lattice of all decompositions. The reader will easily verify that, while $\mathfrak{D}_m(X)$ as a partially ordered set is a lattice, it is not in general, a sublattice of $\mathfrak{D}(X)$.

Define a continuum X to be a *generalized dendrite* if there is a unique irreducible subcontinuum containing each pair of its points. It is easy to prove that X is a generalized dendrite if and only if the intersection of two continua is either a continuum or is vacuous.

THEOREM 3.1. *The continuum X is a generalized dendrite if and only if $\mathfrak{D}_m(X)$ is a sublattice of $\mathfrak{D}(X)$.*

Proof. Since the meet of monotone decompositions is monotone, we need only consider joins. Suppose then that X is a generalized dendrite and D_1 and D_2 are monotone u. s. c. decompositions of it. Their join, $D_1 \vee D_2$, is the decomposition whose slices are the intersections of slices of D_1 and D_2. But these intersections are continua.

Conversely, suppose $\mathfrak{D}_m(X)$ is a sublattice of $\mathfrak{D}(X)$. Then the join of two monotone decompositions is monotone, which implies that the intersection of continua is a continuum. Therefore X is a generalized dendrite.

THEOREM 3.2. *If X is a dendrite, then $\mathfrak{D}_m(X)$ is a distributive sublattice of $\mathfrak{D}(X)$.*

Proof. By a theorem due to S. Bergman, it suffices to show that relative complements are unique [5, p. 75]. This may be done directly, but it is simpler to use Theorems 2.4 and 3.1. Then $\mathfrak{D}_m(X)$ is a sublattice of $\mathfrak{D}(X)$, isomorphic to $\mathfrak{D}_m(A)$. It is clear that the structure of $\mathfrak{D}_m(X)$ as a partially ordered system coincides with its structure as a sublattice. To prove its distributivity it is only necessary to show that $\mathfrak{D}_m(A)$ is distributive, or that relative complements are unique. But this latter is not difficult since A is an arc, and is left to the reader.

A continuum, each of whose subcontinua is indecomposable, will be called a *Knaster continuum.* The only example of such a continuum is due to B. Knaster [3].

THEOREM 3. 3. *If X is a Knaster continuum then $\mathfrak{D}_m(X)$ is a distributive sublattice of $\mathfrak{D}(X)$.*

Proof. A lattice is distributive if and only if it does not contain one of the lattices L_1 or L_2 as a sublattice [5, p. 75].

$$
\begin{aligned}
L_1:\ & D_1 > D_2, D_3, D_4 > D_5 \text{ and}\\
& \text{(a)}\quad D_2 \vee D_3 = D_3 \vee D_4 = D_4 \vee D_2 = D_1,\\
& \text{(b)}\quad D_2 \wedge D_3 = D_3 \wedge D_4 = D_4 \triangle D_2 = D_5.\\
L_2:\ & D_1 > D_2, D_3, D_4 > D_5, \text{ with } D_2 > D_3 \text{ and}\\
& \text{(c)}\quad D_2 \vee D_4 = D_3 \vee D_4 = D_1,\\
& \text{(d)}\quad D_2 \wedge D_4 = D_3 \wedge D_4 = D_5.
\end{aligned}
$$

If C_1 and C_2 are subcontinua of a Knaster continuum with a non-empty intersection, then either $C_1 \supset C_2$ or $C_2 \supset C_1$ [3, p. 281]. This implies that the Knaster continuum is a generalized dendrite.

Let S_i, $(i = 1, \cdots, 5)$, be the slices of D_i containing an arbitrary point x. Observe that because of the above property of Knaster continua, to get the meet of two decompositions, one merely chooses the decomposition whose slice containing an arbitrary point x is the greater of the two slices. It is easy to see that this is an u. s. c. decomposition.

Consider L_1. From the ordering of the D_i it follows that $S_1 \subset S_2, S_3, S_4 \subset S_5$, and from (a) that $S_2 \cdot S_3 = S_1$, so either $S_2 \subseteq S_3$ or $S_3 \subseteq S_2$. Suppose the former, then $S_1 = S_2$. But $S_3 \cdot S_4 = S_1 = S_2$, and so $S_4 = S_1 = S_2$. It follows from (b) and the manner in which meets are formed that $S_5 = S_2 + S_3 = S_3 = S_3 + S_4 = S_2 + S_4 = S_2 = S_4$. Thus, $S_i = S_j$, $(i, j = 1, \cdots, 5)$. Since this holds for the slices containing an arbitrary point, $D_i = D_j$, $(i, j = 1, \cdots, 5)$.

Now consider L_2. With the same notation for slices, we have $S_1 \subset S_2, S_3, S_4 \subset S_5$ and $S_2 \subset S_3$. Then from (c), $S_1 = S_2 \cdot S_4 = S_3 \cdot S_4$ and there are three cases to be considered.

$$
\begin{aligned}
&(1)\quad S_2 \subset S_4 \subset S_3.\\
&(2)\quad S_2 \subset S_3 \subset S_4.\\
&(3)\quad S_4 \subset S_2 \subset S_3.
\end{aligned}
$$

In case (1) we get from (d) that $S_4 + S_3 = S_5 = S_3 = S_4 + S_2 = S_4$, and from (c) that $S_2 \cdot S_4 = S_1 = S_2 = S_3 \cdot S_4 = S_4$. Thus $S_2 = S_3$.

In case (2) we get from (c) that $S_2 \cdot S_4 = S_1 = S_2 = S_3 \cdot S_4 = S_3$. Thus $S_2 = S_3$.

In case (3) we get from (d) that $S_3 + S_4 = S_3 = S_5 = S_2 + S_4 = S_2$. Thus $S_2 = S_2$.

So in all cases $S_2 = S_3$ and the two relative complements are equal.

Consider now the system, $\mathfrak{D}_{1m}(X)$, of all monotone decompositions of the continuum X, u. s. c. or not. It is almost immediate that X is hereditarily locally connected if and only if all monotone decompositions are u. s. c., or $\mathfrak{D}_{1m}(X) = \mathfrak{D}_m(X)$. For if there is a continuum of convergence, C, in X, $\lim C_n = C$, then the slices C_n define a monotone decomposition which is not upper semi-continuous. Conversely, if all monotone decompositions are u. s. c., then every sequence of continua is a null sequence and there are no continua of convergence.

THEOREM 3. 4. *The continuum X is a dendrite if and only if $\mathfrak{D}_m(X)$ is a sublattice of $\mathfrak{D}(X)$ and one of the following conditions is satisfied*:

(1) $\mathfrak{D}_{1m}(X) = \mathfrak{D}_m(X)$;
(2) *$\mathfrak{D}_m(X)$ is isomorphic to $\mathfrak{D}_m(A)$, where A is an arc*;
(3) *Simple monotone decompositions have complements in $\mathfrak{D}_m(X)$.*

Proof. The sublattice condition makes X a generalized dendrite and (1), (2), and (3) each imply the hereditary local connectivity of X. The only statement to verify is (3). If X were to have a continuum of convergence, C, then it is easy to see that the simple decomposition having C as its non-degenerate slice has no complement.

4. Examples and problems. On the unit interval, $(0, 1)$, denote by E the usual Cantor set, obtained by deleting middle thirds. The closures of these deleted sub-intervals define a monotone decomposition, D_E, of $(0, 1)$, which will be called the Cantor decomposition. Since D_E has no complement, $\mathfrak{D}_m((0, 1))$ is not a Boolean algebra.

It has been shown that $\mathfrak{D}_m(X)$ is distributive for dendrites and Knaster continua, two extreme types. Thus the distributivity of $\mathfrak{D}_m(X)$ need not imply anything about the local connectivity of X. However in the simpler non-locally connected continua $\mathfrak{D}_m(X)$ need not even be modular. Suppose X contains an arc A_0 which is the limit of the distinct arcs A_i, $(i = 1, 2, \cdots)$. For example, X might be the familiar sinusoid. Consider, for simplicity, each arc A_i as the unit interval $(0, 1)$, with points on A_i with coördinate a, having as a limit the point on A_0 with coördinate a.

In forming the Cantor decomposition there have been removed after the n-th stage $2^n - 1$ middle thirds. The remaining intervals define a monotone

decomposition D_n'. The closures of the removed intervals define a monotone decomposition D_n''. $D_n' \vee D_n'' = I$, and $D_n' \wedge D_n'' = 0$ on A_n.

Let D_4 be a monotone decomposition of X which coincides with D_n' on A_n, $(n = 1, 2, \cdots)$, and has degenerate slices everywhere else. Let D_2 be the monotone decomposition which coincides with D_n'' on A_n, $(n = 1, 2, \cdots)$, and with the Cantor decomposition on A_0. Let D_3 be the monotone decomposition which coincides with D_n'' on A_n, $(n = 1, 2, \cdots)$, and has A_0 as a single slice. All other slices are points. Let D_5 be the monotone decomposition which has exactly A_n, $(n = 0, 1, 2, \cdots)$, as its non-degenerate slices, and let $D_1 = I$. It is easy to see that D_1, D_2, D_3, D_4, and D_5 form the lattice L_2 of the proof of Theorem 3. 3, and hence $\mathfrak{D}_m(X)$ is not modular.

The following problems suggest themselves.

(1) Give a lattice characterization of $\mathfrak{D}_m(A)$.

(2) How does the lattice structure of $\mathfrak{D}_m(X)$, where X is a Knaster continuum, differ from that of $\mathfrak{D}_m(A)$?

(3) Are there continua for which $\mathfrak{D}_m(X)$ is modular but not distributive?

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AXISYMMETRIC HARMONIC VECTORS.*[1]

By Morris Marden.

1. Introduction. As is well-known, the theory of analytic functions of a complex variable provides an important method for the study of harmonic functions of two real variables. As shown by Bergman[2] the theory of analytic functions may also be used in the study of harmonic functions of three real variables as well as "harmonic vectors" $\boldsymbol{H}(x, y, z)$ i. e. triples $H_1(x, y, z)$, $H_2(x, y, z)$, $H_3(x, y, z)$ of harmonic functions such that the vector

$$\boldsymbol{H}(x, y, z) = H_1(x, y, z)\boldsymbol{i}_x + H_2(x, y, z)\boldsymbol{i}_y + H_3(x, y, z)\boldsymbol{i}_z$$

satisfies the two relations

$$\nabla \cdot \boldsymbol{H} = 0, \qquad \nabla \times \boldsymbol{H} = 0. \tag{1.1}$$

The essential feature of Bergman's approach is the use of his operator (a generalization of the Whittaker formula):

$$P(f, \Gamma) = P(f, \Gamma, T) = (2\pi i)^{-1} \int_\Gamma f(u, \zeta)\zeta^{-1} d\zeta. \tag{1.2}$$

Here ζ is a complex variable,

$$u = x + (i/2)[(y + iz)\zeta^{-1} + (y - iz)\zeta], \tag{1.3}$$

Γ is a closed, rectifiable curve in the ζ-plane and (x, y, z) represents a point in a sufficiently small neighborhood T of some given point (x_0, y_0, z_0). Let S be a region in the u-plane which contains the locus of points u satisfying relation (1.3) when the point (x, y, z) is in T and the point ζ is on Γ. The

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[2] S. Bergman, *Mathematische Zeitschrift*, vol. 24 (1926), pp. 641-669; *Mathematische Annalen*, vol. 99 (1928), pp. 629-658 and vol. 100 (1929), pp. 534-558; *Bulletin of the American Mathematical Society*, vol. 49 (1943), pp. 163-174. See also E. P. Beckenbach, *Bulletin of the American Mathematical Society*, vol. 48 (1942), pp. 937-941, in which harmonic vectors are called "Newtonian" vectors.

operator $P(f, \Gamma, T)$ transforms the class of functions $f(u, \zeta)$ which are analytic in u for u in S and continuous in ζ for ζ on Γ, into the class of complex functions $H_1(x, y, z)$ which are harmonic in T. As proved by Bergman, the harmonic functions

$$(1.4) \qquad \begin{aligned} H_2(x, y, z) &= P[(i/2)(\zeta + \zeta^{-1})f, \Gamma, T], \\ H_3(x, y, z) &= P[2^{-1}(\zeta - \zeta^{-1})f, \Gamma, T], \end{aligned}$$

together with $H_1(x, y, z)$ have the properties that $H_1(x, y, z)dx + H_2(x, y, z)dy + H_3(x, y, z)dz$ is a total differential and that the vector $\boldsymbol{H}(x, y, z) = H_1(x, y, z)\boldsymbol{i}_x + H_2(x, y, z)\boldsymbol{i}_y + H_3(x, y, z)\boldsymbol{i}_z$ satisfies the two relations (1.1).

An important subclass of harmonic functions consists of the axisymmetric harmonic functions $\Phi(x, \rho)$, where $\rho^2 = y^2 + z^2$. Such functions may be generated by the operation $P(f, \Gamma, T)$ applied to the function $f(u, \zeta) = \phi(u)$, where $\phi(u)$ is analytic in S. As an axisymmetric harmonic function, $\Phi(x, \rho)$ may be regarded as the velocity potential of a flow symmetric in the x-axis and thus to it corresponds a so-called Stokes' Stream Function $\Psi(x, \rho)$.[3] As we shall see in the following section,

$$(1.5) \qquad \Psi(x, \rho)^2 = \rho^2 H_2(x, y, z)^2 + \rho^2 H_3(x, y, z)^2,$$

where $H_2(x, y, z)$ and $H_3(x, y, z)$ are the functions obtained from formulas (1.4) when $f(u, \zeta) = \phi(u)$.

We are thus led to consider the axisymmetric harmonic vector

$$(1.6) \qquad \boldsymbol{H}(x, \rho) = \Phi(x, \rho)\boldsymbol{i}_x - \rho^{-1}\Psi(x, \rho)\boldsymbol{i}_\rho$$

where $\boldsymbol{i}_x$ and $\boldsymbol{i}_\rho$ are the unit vectors along the x and ρ-axes in the meridian plane. In the following sections we shall study the properties of the vector $\boldsymbol{H}(x, \rho)$ in relation to the properties of the analytic function $\phi(u)$ *associated* with $\boldsymbol{H}(x, \rho)$. Physically speaking, our problem is the study of the axisymmetric flows represented by the vector $\boldsymbol{H}(x, \rho)$ in relation to the two-dimensional flows represented by the *associate function* $\phi(u)$.

Thus, in **2**, we shall express $\boldsymbol{H}(x, \rho)$ in terms of $\phi(u)$ and, conversely, $\phi(u)$ in terms of $\boldsymbol{H}(x, \rho)$. In **3**, we shall study the $\boldsymbol{H}(x, \rho)$ corresponding to a $\phi(u)$ having a pole of order n; the corresponding $\boldsymbol{H}(x, \rho)$ has, in general, a branch line along a circle. In **4**, for the purpose of constructing a flow of more practical interest, we shall consider the $\phi(u)$ obtained by superposing

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[3] In general, both $\Phi(x, \rho)$ and $\Psi(x, \rho)$ will be complex functions and there is thus an axisymmetric flow whose potential and stream-functions are respectively $\mathfrak{R}(\Phi)$ and $\mathfrak{R}(\Psi)$, and another such flow with these functions as $\mathfrak{J}(\Phi)$ and $\mathfrak{J}(\Psi)$.

certain continuous distributions of poles (multiple sources and multiple vortices). The corresponding $\boldsymbol{H}(x,\rho)$ then represents the flow due to a superposition of multiple source filaments and multiple vortex filaments. We study, in **5**, the $H(x,\rho)$ corresponding to essential singularities in $\phi(u)$ and, in **6**, the $\boldsymbol{H}(x,\rho)$ corresponding to periodic $\phi(u)$. Finally, in **7**, we compute the energy in the field due to any branch of $\boldsymbol{H}(x,\rho)$ in terms of the transforms of the energy in the field with force potential $\phi(u)$.

2. The vector $\boldsymbol{H}(x,\rho)$ and the associated analytic function $\varphi(u)$. We shall begin by computing $H_1(x,y,z)$, when $f(u,\zeta)=\phi(u)$, where $\phi(u)$ is analytic in a ρ-convex region S; i. e., a region S which, if it contains a point (x,ρ), contains also all of the points $(x,\lambda\rho)$ with $-1\leqq\lambda\leqq 1$.

$$H_1(x,y,z)=P[\phi(u),|\zeta|=1]=(2\pi)^{-1}\int_0^{2\pi}\phi(x+iy\cos t+iz\sin t)\,dt$$
$$=(2\pi)^{-1}\int_{0-\alpha}^{2\pi-\alpha}\phi(x+i\rho\cos t)\,dt,$$

where $y=\rho\cos\alpha$ and $z=\rho\sin\alpha$. Hence,

$$H_1(x,y,z)=(2\pi)^{-1}\int_0^{2\pi}\phi(x+i\rho\cos t)\,dt=\Phi(x,\rho). \tag{2.1}$$

Next, let us evaluate $H_2(x,y,z)\boldsymbol{i}_y+H_3(x,y,z)\boldsymbol{i}_z$.

$$\begin{aligned}H_2\boldsymbol{i}_y+H_3\boldsymbol{i}_z&=P[(i/2)(\zeta+\zeta^{-1})\phi(u),|\zeta|=1]\boldsymbol{i}_y\\&\quad+P[(1/2)(\zeta-\zeta^{-1})\phi(u),|\zeta|=1]\boldsymbol{i}_z\\&=i(2\pi)^{-1}\int_0^{2\pi}\phi(x+iy\cos t+iz\sin t)(\boldsymbol{i}_y\cos t+\boldsymbol{i}_z\sin t)\,dt.\end{aligned}$$

Expanding $\cos(t+\alpha)$ and $\sin(t+\alpha)$, using the fact that $\int_0^{2\pi}\phi(u)\sin t\,dt=0$ and substituting $\boldsymbol{i}_\rho=\boldsymbol{i}_y\cos t+\boldsymbol{i}_z\sin t$, we find [4]

$$H_2\boldsymbol{i}_y+H_3\boldsymbol{i}_z=i(2\pi)^{-1}\boldsymbol{i}_\rho\int_0^{2\pi}\phi(x+i\rho\cos t)\cos t\,dt=-\rho^{-1}\Psi(x,\rho)\boldsymbol{i}_\rho \tag{2.2}$$

and thus

$$\boldsymbol{H}(x,\rho)=\Phi\boldsymbol{i}_x-\rho^{-1}\Psi\boldsymbol{i}_\rho=(2\pi)^{-1}\int_0^{2\pi}\phi(x+i\rho\cos t)(\boldsymbol{i}_x+i\,\boldsymbol{i}_\rho\cos t)\,dt.$$

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[4] Shortly after our first announcement of this formula (*Bulletin of the American Mathematical Society*, vol. 49 (1943), p. 694, abstract 225), the formula appeared in print in the paper by L. Bers and A. Gelbart in *Quarterly Journal of Applied Mathematics*, vol. 1 (1943), pp. 168-188.

We shall now show that $\Psi(x,\rho)$, as defined by formula (2.2), is the Stokes' Stream Function corresponding to the velocity potential $\Phi(x,\rho)$ given by formula (2.1). For this purpose, it is necessary and sufficient to show that $\Phi(x,\rho)$ and $\Psi(x,\rho)$ together satisfy the system of differential equations:

$$(2.3)\quad \nabla\cdot\boldsymbol{H}=\rho^{-1}\partial\Psi/\partial\rho-\partial\Phi/\partial x=0,\quad \nabla\times\boldsymbol{H}=(\rho^{-1}\partial\Psi/\partial x+\partial\Phi/\partial x)\boldsymbol{p}=\boldsymbol{0}$$

where $\boldsymbol{p}$ is a unit vector perpendicular to the meridian plane.

By differentiating equation (2.2) and then integrating by parts one of the resulting integrals, we find

$$\partial\Psi/\partial\rho=-i(2\pi)^{-1}\int_0^{2\pi}\phi(x+i\rho\cos t)+\rho(2\pi)^{-1}\int_0^{2\pi}\phi'(x+i\rho\cos t)\cos^2 t\,dt$$

$$=\rho(2\pi)^{-1}\int_0^{2\pi}\phi'(x+i\rho\cos t)(\cos^2 t+\sin^2 t)dt=\rho(\partial\Phi/\partial x).$$

Similarly, we may verify that $\nabla\times\boldsymbol{H}=\boldsymbol{0}$.

The above results may be summarized as follows.

THEOREM I. *If $\phi(u)$ is analytic in a ρ-convex region S of the meridian plane, then $\boldsymbol{H}(x,\rho)$, defined by the equations*

$$(2.4)\quad \boldsymbol{H}(x,\rho)=(2\pi)^{-1}\int_0^{2\pi}\phi(u)\nabla u\,dt,\quad u=x+i\rho\cos t,\quad \nabla=\boldsymbol{i}_x\partial/\partial x+\boldsymbol{i}_\rho\partial/\partial\rho,$$

is an axisymmetric harmonic vector in the region obtained by revolving S about the x-axis. The functions $\Phi(x,\rho)$ and $\Psi(x,\rho)$, the "components" of $\boldsymbol{H}(x,\rho)$ according to the formula

$$(2.5)\qquad \boldsymbol{H}(x,\rho)=\Phi(x,\rho)\boldsymbol{i}_x-\rho^{-1}\Psi(x,\rho)\boldsymbol{i}_\rho,$$

are, respectively, the velocity potential and Stokes' Stream Function in a three dimensional flow symmetric in the x-axis.[3]

Since $\int_0^{2\pi}\phi(u)\sin t\,dt=0$, a formula alternative to (2.4) is

$$(2.6)\qquad \boldsymbol{H}(x,\rho)=(2\pi)^{-1}\int_0^{2\pi}\phi(u)(\boldsymbol{i}_x+ie^{it}\boldsymbol{i}_\rho)dt$$

If, in Theorem I, the region S is taken as the interior and circumference of the circle $r=a$, where $r^2=x^2+\rho^2$, we may write $\phi(u)=\Sigma c_k u^k$ in the circle $r=a$ and then, using the Laplace integrals for the Legendre polynomials $P_n(\cos\theta)$, we find

$$(2.6)\quad \boldsymbol{H}(x,\rho)=(2\pi)^{-1}\int_0^{2\pi}\sum_{k=0}^{\infty}c_k u^k \nabla u\,dt$$
$$=\sum_{k=0}^{\infty}c_k r^k[P_k(x/r)\boldsymbol{i}_x+i(\rho/r)P'_k(x/r)\boldsymbol{i}_\rho].$$

Conversely, in order to determine $\phi(u)$ when $\boldsymbol{H}(x,\rho)$ is given, let us use the orthogonality property of the Legendre functions. Thus,

$$\int_0^\pi \boldsymbol{H}(a\cos\theta, a\sin\theta)\cdot[P_n(\cos\theta)\boldsymbol{i}_x-4i\sin\theta P'_n(\cos\theta)\boldsymbol{i}_\rho]d\cos\theta$$
$$=2(2n+1)a^n c_n.$$

Hence, if we define the vector

$$\boldsymbol{P}_n(\cos\theta)=(4n+2)^{-1}[P_n(\cos\theta)\boldsymbol{i}_x-4i\sin\theta\,P'_n(\cos\theta)\boldsymbol{i}_\rho]$$

we may write

$$(2.7)\qquad c_n=a^{-n}\int_0^\pi \boldsymbol{H}(a\cos\theta,\ a\sin\theta)\cdot\boldsymbol{P}_n(\cos\theta)\,d\cos\theta.$$

If $\boldsymbol{H}(x,\rho)$ is harmonic in $r\leqq a$, it is continuous and therefore its magnitude is bounded in $r\leqq a$. From the properties of Legendre polynomials, it follows that the magnitude of $\boldsymbol{P}_n$ is also bounded. Hence, the $a^n c_n$ are also bounded and the series $\phi(u)=\Sigma c_n u^n$ represents an analytic function at points interior to the circle $r=a$.

We have thus obtained the following result:

THEOREM II. *Let $\boldsymbol{H}(x,\rho)$ be an axisymmetric vector which is harmonic in and on the sphere $x^2+\rho^2=a^2$. Let $P_n(\cos\theta)$ denote the Legendre polynomials of degree n and let*

$$\boldsymbol{P}(u,\theta)=\sum_{n=0}^{\infty}(4n+2)^{-1}(u/a)^n[P_n(\cos\theta)\boldsymbol{i}_x-4i\sin\theta P'_n(\cos\theta)\boldsymbol{i}_\rho].$$

Then an associated analytic function corresponding to $H(x,\rho)$ for points (x,ρ) within the sphere is the function

$$\phi(x+i\rho)=\int_0^\pi \boldsymbol{H}(a\cos\theta,\ a\sin\theta)\cdot\boldsymbol{P}(x+i\rho,\theta)\,d\cos\theta.$$

3. Associate function with a pole. In the following we shall mean by a ρ-convex neighborhood $N_\rho(a,b)$ of a point $u=x+i\rho=a+ib$, $b>0$, the smallest ρ-convex region enclosing the circle $|u-a-ib|=\epsilon$ and by a ρ-convex neighborhood $N'_\rho(a,b)$ that obtained by cutting $N_\rho(a,b)$ along the line segments $x=a$, $b\leqq|\rho|\leqq b+\epsilon$.

If $\phi(x+i\rho)$ is analytic in $N'_\rho(a,b)$ and has a pole of order n at $x+i\rho=a+ib$, then we may write $\phi(u)$ in the form

$$\phi(u)=\sum_1^n c_k(x+i\rho-a-bi)^{-k}+f(x+i\rho) \tag{3.1}$$

where $f(u)$ is analytic throughout $N_\rho(a,b)$. Formally, the corresponding harmonic vector is

$$\boldsymbol{H}(x,\rho)=\sum_{k=1}^n \boldsymbol{H}_k(x,\rho)+\boldsymbol{F}(x,\rho) \tag{3.2}$$

where

$$\boldsymbol{H}_k(x,\rho)=(2\pi)^{-1}\int_0^{2\pi}(u-a-ib)^{-k}\nabla u\,dt,\quad u=x+i\rho\cos t, \tag{3.3}$$

$$\boldsymbol{F}(x,\rho)=(2\pi)^{-1}\int_0^{2\pi}f(u)\nabla u\,dt.$$

Setting $w=e^{it}$ in formula (2.6) and writing

$$2w(x+i\rho\cos t-a-bi)=i\rho(w-w_1)(w-w_2)$$

where

$$w_k=i\rho^{-1}\{x-a-bi+(-1)^k[(x-a-ib)^2+\rho^2]^{1/2}\}, \tag{3.4}$$

we find

$$\boldsymbol{H}_1(x,\rho)=-\pi\rho^{-1}\int_C(w-w_1)^{-1}(w-w_2)^{-1}(\boldsymbol{i}_x+iw\boldsymbol{i}_\rho)dw, \tag{3.5}$$

where C is the unit circle $|w|=1$. Each function $w_k(x,\rho)$ is single-valued in $N'_\rho(a,b)$, and, as $w_1w_2=1$, one of the points w_1, w_2 is inside the unit circle $|w|=1$ and the other is outside this circle, unless both are coincident on this circle. The latter can occur only when the point (x,ρ) is on the radii cut from $N_\rho(a,b)$ in producing $N'_\rho(a,b)$. Let us introduce two sheets over $N'_\rho(a,b)$ and consider the points (x,ρ) of the first sheet to correspond to $|w_1|<1$ and $|w_2|>1$, whereas points (x,ρ) in the second sheet correspond to $|w_1|>1$ and $|w_2|<1$. Then, according to the residue theorem, the value of $\boldsymbol{H}_1(x,\rho)$ on the first sheet is

$$\boldsymbol{H}_1^{(1)}(x,\rho)=2i\rho^{-1}(w_2-w_1)^{-1}(\boldsymbol{i}_x+iw_1\boldsymbol{i}_\rho)$$

and on the second sheet

$$\boldsymbol{H}_1^{(2)}(x,\rho)=2i\rho^{-1}(w_1-w_2)^{-1}(\boldsymbol{i}_x+iw_2\boldsymbol{i}_\rho).$$

From these formulae, since.

$$\boldsymbol{H}_n(x,\rho) = (-1)^{n-1}[(n-1)!]^{-1}\,\partial^{n-1}\boldsymbol{H}_1(x,\rho)/\partial x^{n-1},$$

we may obtain the corresponding ones for $\boldsymbol{H}_n(x,\rho)$. In particular, on substituting the values of w_1 and w_2 from (3.4) and setting $x + i\rho - a - ib = \lambda e^{i\theta}$, we learn that for $e^{i\theta} \neq \pm i$ and for $\lambda \to 0$,

$$|\boldsymbol{H}_1(x,\rho)| = O(\lambda^{-\frac{1}{2}}) \text{ or } O(\lambda^{-1}) \text{ according as } b \neq 0 \text{ or } b = 0.$$

Thus

$$|\boldsymbol{H}_n(x,\rho)| = O(\lambda^{\frac{1}{2}-n}) \text{ or } O(\lambda^{-n}) \text{ according as } b \neq 0 \text{ or } b = 0.$$

We may summarize our results as follows.

THEOREM III. *Let $\phi(u)$ be analytic in the neighborhood $N'_\rho(a,b)$ and have a pole of order n at $u = a + ib$. Then the corresponding vector $\boldsymbol{H}(x,\rho)$ is two-valued and harmonic in the region $R[N'_\rho(a,b)]$ obtained on rotating the region $N'_\rho(a,b)$ about the x-axis and has as branch line the circle $x = a$, $\rho = b$. Along this circle, $|\boldsymbol{H}(x,\rho)|$ behaves like $[(x-a)^2 + (\rho-b)^2]^{-\frac{1}{2}(n-\frac{1}{2})}$ when $b > 0$, but like $[(x-a)^2 + \rho^2]^{-n/2}$ when $b = 0$.*

4. Circular source and vortex filaments. As is well-known, the principal part of a function $\phi(u)$, having a pole of order n at the point $u = x + i\rho = \xi + i\eta = \zeta$, may be regarded as the complex velocity potential of the two-dimensional flow due to the superposition at $u = \zeta$ of point sources and point vortices of multiplicities from 2 to $n + 1$. According to Theorem III, when the associated function $\phi(u)$ has an n-th order pole on the x-axis, the corresponding $\boldsymbol{H}(x,\rho)$ has an n-th order pole at the same point on the x-axis, but, when $\phi(u)$ has a pole at (ξ,η) not on the x-axis, $\boldsymbol{H}(x,\rho)$ has along the circle $x = \xi$, $\rho = \eta$ a line singularity which does not have the character of a multiple circular source or vortex filament. We shall now show that a harmonic vector with a source or vortex filament type of singularity can be derived essentially by two successive applications of the operation (2.4) or (2.6).

More precisely, let us denote the operation (2.4) which transforms a given analytic function $\phi(u,\zeta)$ of $u = x + i\rho$ and $\zeta = \xi + i\eta$ into axisymmetric harmonic vectors thus:

$$(4.1)\quad \boldsymbol{A}_u[\phi(u,\zeta)] = (2\pi)^{-1}\int_0^{2\pi}\phi(u,\zeta)\,\nabla u\,dt,\ u = x + i\rho\cos t,\ \nabla = \boldsymbol{i}_x\partial/\partial x + \boldsymbol{i}_\rho\partial/\partial\rho$$

$$(4.2)\quad \boldsymbol{A}_\zeta[\phi(u,\zeta)] = (2\pi)^{-1}\int_0^{2\pi}\phi(u,\zeta)\,\nabla\zeta\,d\alpha,\ \zeta = \xi + i\eta\cos\alpha,\ \nabla = \boldsymbol{i}_x\partial/\partial\xi + \boldsymbol{i}_\rho\partial/\partial\eta.$$

We shall now show that the harmonic vector

(4.3) $$S_1(x, \rho) = \lambda_1 A_u\{i_x \cdot A_\zeta (u - \zeta)^{-1}\}, \qquad \lambda_1 \text{ real},$$

represents the flow due to a circular source filament of total strength λ_1, along the circle $x = \xi$, $\rho = \eta$, and that the harmonic vector

(4.4) $$V_1(x, \rho) = \mu_1 A_u\{i_\rho \cdot A_\zeta (u - \zeta)^{-1}\}, \qquad \mu_1 \text{ real},$$

represents the flow due to a circular vortex filament of total strength μ_1 along the same circle $x = \xi$, $\rho = \eta$.

To prove the first statement, let us compute the potential function (Cf. Theorem I)

(4.5) $$\begin{aligned} i_x \cdot S_1(x, \rho) &= \lambda_1 i_x \cdot A_u\{i_x \cdot A_\zeta (u - \zeta)^{-1}\} \\ &= \lambda_1 (2\pi)^{-2} \int_0^{2\pi} dt \int_0^{2\pi} (x + i\rho \cos t - \xi - i\eta \cos \alpha)^{-1} d\alpha \\ &= \lambda_1 (2\pi)^{-2} \int_0^{2\pi} dt \int_0^{2\pi} [x + i\rho \cos t - \xi - i\eta \cos(\alpha + t)]^{-1} d\alpha \\ &= \lambda_1 (2\pi)^{-2} \int_0^{2\pi} d\alpha \int_0^{2\pi} [x - \xi + i(\rho - \eta \cos \alpha) \cos t + i\eta \sin \alpha \sin \\ &= \lambda_1 (2\pi)^{-1} \int_0^{2\pi} [(x - \xi)^2 + (\rho - \eta \cos \alpha)^2 + \eta^2 \sin^2 \alpha]^{-1/2} dt. \end{aligned}$$

This is precisely the potential due to a circular source filament of total strength λ_1 along the circle $x = \xi$, $\rho = \eta$.

Likewise, let us compute the Stokes' stream function (Cf. Theorem I)

(4.6) $$\begin{aligned} -\rho i_\rho \cdot V_1(x, \rho) &= -\rho \mu_1 i_\rho \cdot A_u\{i_\rho \cdot A_\zeta (u - \zeta)^{-1}\} \\ &= -\rho \mu_1 (2\pi)^{-2} \int_0^{2\pi} dt \int_0^{2\pi} [x + i\rho \cos t - i\eta \cos \alpha - \xi]^{-1} \cos(\alpha - t)\, d\alpha \\ &= -\rho \mu_1 (2\pi)^{-2} \int_0^{2\pi} dt \int_0^{2\pi} [x + i\rho \cos t - \xi - i\eta \cos(\alpha + t)]^{-1} \cos \alpha\, d\alpha \\ &= -\rho \mu_1 (2\pi)^{-1} \int_0^{2\pi} [(x - \xi)^2 + (\rho - \eta \cos \alpha)^2 + \eta^2 \sin^2 \alpha]^{-1/2} \cos \alpha\, d\alpha. \end{aligned}$$

This is precisely the Stokes' stream function due to a circular vortex filament of total strength μ_1.

It is to be noted that in the elliptic integrals (4.5) and (4.6) we restrict the points (x, ρ) to the neighborhood $N'_\rho(\xi, \eta)$ and the points (ξ, η) to the neighborhood $N'_\eta(x, \rho)$.

Combining the above results, we see that the harmonic vector

(4.7) $$H_1(x, \rho) = A_u(\lambda_1 i_x + \mu_1 i_\rho) \cdot A_\zeta (u - \zeta)^{-1}$$

represents the flow due to the superposition along the circle $x = \xi$, $\rho = \eta$ of a circular source filament of total strength λ_1, and of a circular vortex filament of total strength μ_1.

Let us call a multiple source or vortex point or filament, having a multiplicity of n and having an axis in the direction of the positive x-axis, an *x-oriented n-source* or *n-vortex point* or *filament.* Since the complex velocity potential or harmonic vector for an n-source or n-vortex is the $(n-1)$-st derivative with respect to x of the potential or harmonic vector for a simple source or vortex, we deduce at once, from the preceding discussion, the following theorem:

THEOREM IV. *The axisymmetric harmonic vector $\boldsymbol{H}_n(x, \rho)$ representing the flow due to an x-oriented n-source filament of total strength λ_n superposed along the circle $x = \xi$, $\rho = \eta$ upon an x-oriented n-vortex filament of total strength μ_n is given by the formula*:

$$\text{(4. 8)} \qquad \boldsymbol{H}_n(x, \rho) = (-1)^{n-1}[(n-1)!]\boldsymbol{A}_u(\lambda_n \boldsymbol{i}_x + \mu_n \boldsymbol{i}_\rho) \cdot \boldsymbol{A}_\zeta(u-\zeta)^{-n}.$$

It is to be noted that this result is analogous to the well-known expression

$$\phi_n(u) = (-1)^{n-1}(n-1)!(\lambda_n + i\mu_n)(u-\zeta)^{-n}.$$

for the complex velocity potential of the two-dimensional flow due to an x-oriented $(n+1)$-source point superposed at $u = \zeta$ upon an x-oriented $(n+1)$-vortex point. Furthermore, in formula (4. 8), if $\eta = 0$, we deduce $\boldsymbol{H}_n(x, \rho) = (-1)^{n-1}(n-1)!\boldsymbol{A}_u\lambda_n(u-\xi)^{-n}$, the result given at the close of Theorem III.

Since we may write (4. 8) as $\boldsymbol{H}_n(x, \rho) = (-1)^{n-1}(n-1)!\boldsymbol{A}_u\phi_n(u, \zeta)$ where

$$\begin{aligned}\phi_n(u, \zeta) &= (2\pi)^{-1}\int_0^{2\pi}(\lambda_n + i\mu_n \cos\alpha)(u-\xi-i\eta\cos\alpha)^{-1}d\alpha \\ &= \int_{-\eta}^{+\eta}[\pi^{-1}\lambda_n\eta(\eta^2-h^2)^{-\frac{1}{2}} + i\pi^{-1}\mu_n(\eta^2-h^2)^{-\frac{1}{2}}](u-\xi-ih)^{-1}dh,\end{aligned}$$

we may restate Theorem IV in terms of the complex velocity potential due to certain continuous distributions of multiple point sources and vortices along the line segment $x = \xi$, $|\rho| \leqq \eta$.

THEOREM IV′. *Let $\boldsymbol{H}_n(x, \rho)$ be the axisymmetric harmonic vector representing the flow due to an x-oriented n-source filament of total strength λ_n superposed along the circle $x = \xi$, $\rho = \eta$ upon an x-oriented n-vortex filament of total strength μ_n. Then the analytic function $\phi_n(x + i\rho)$ which is the asso-*

ciate of $\boldsymbol{H}_n(x,\rho)$ *is the complex velocity potential due to a continuous distribution of x-oriented* $(n+1)$*-source points of strength* $\pi^{-1}\lambda_n\eta(\eta^2-h^2)^{-\frac{1}{2}}$ *per unit length superposed along the line segment* $x=\xi$, $|\rho|\leqq\eta$ *upon a continuous distribution of x-oriented* $(n+1)$*-vortex points of strength* $\pi^{-1}\mu_n(\eta^2-h^2)^{-\frac{1}{2}}$ *per unit length.*

Finally, from the principal part $\sum_1^n(\lambda_k+i\mu_k)(u-\zeta)^{-k}$ of a function analytic in the neighborhood of $u=\zeta$ except for an n-th order pole at $u=\zeta$, we may derive an analogous "principal part" for an axisymmetric vector which is harmonic in the neighborhood $N'_\rho(\xi,\eta)$ and has a "pole" along the $x=\xi$, $\rho=\eta$; namely,

$$\boldsymbol{A}_u\sum_{k=1}^n(\lambda_k\boldsymbol{i}_x+\mu_k\boldsymbol{i}_\rho)\cdot\boldsymbol{A}_\zeta(u-\zeta)^{-k}.$$

The latter is a harmonic vector which represents the flow due to the superposition along the circle $x=\xi$, $\rho=\eta$ of x-oriented k-source and k-vortex filaments of strengths λ_k and μ_k, where $k=1,2,\cdots,n$.

5. An isolated essential singularity. Let $F(u,\alpha)$ be a function which is continuous in α for $0\leqq\alpha\leqq 2\pi$ and which for each α is analytic in the neighborhood

$$N(\alpha):\quad 0<\delta\leqq|u-\xi-i\eta\cos\alpha|\leqq\Delta$$

of the point $u=\xi+i\eta\cos\alpha$, where $F(u,\alpha)$ is assumed to have an isolated essential singularity. Let us suppose also uniform convergence with respect to both u and α for the infinite series portion of the Laurent development of $F(u,\alpha)$:

$$F(u,\alpha)=\sum_{n=1}^{\infty}(a_n+ib_n\cos\alpha)(u-\xi-i\eta\cos\alpha)^{-n}+f(u,\alpha)\tag{5.1}$$

where $f(u,\alpha)$ is analytic throughout the circle $|u-\xi-i\eta\cos\alpha|\leqq\Delta$. Let us assume further that a_n and b_n are real constants, independent of u and α.

Then

$$\phi(u)=(2\pi)^{-1}\int_0^{2\pi}F(u,\alpha)d\alpha=\sum_{n=1}^{\infty}(a_n\boldsymbol{i}_x+b_n\boldsymbol{i}_\rho)\cdot\boldsymbol{A}_\zeta(u-\zeta)^{-n}+f_1(u)$$

is single-valued and analytic in any closed ρ-convex region $N''_\rho(\xi,\eta)$ contained in the neighborhood $N'_\rho(\xi,\eta)$ and the vector

$$\boldsymbol{H}(x,\rho)=\sum_{n=1}^{\infty}\boldsymbol{A}_u(a_n\boldsymbol{i}_x+b_n\boldsymbol{i}_\rho)\cdot\boldsymbol{A}_\zeta(u-\zeta)^{-n}+\boldsymbol{A}_uf_1(u)\tag{5.2}$$

is single-valued and harmonic in the region $R[N''_\rho(\xi,\eta)]$, obtained by re-

volving $N''_\rho(\xi,\eta)$ about the x-axis. Further, in the torus $0 < \delta \leqq (x-\xi)^2 + (\rho-\eta)^2 \leqq \Delta$, the flow represented by $\boldsymbol{H}(x,\rho)$ will be essentially the flow due to the superposition along the circle $x=\xi$, $\rho=\eta$ of the convergent series of x-oriented n-source and n-vortex filaments represented by the "principal part" of $\boldsymbol{H}(x,\rho)$ in (5.2).

6. Periodic $\boldsymbol{H}(x,\rho)$. For each α, $0 \leqq \alpha \leqq 2\pi$, let $f(u,\alpha)$, $u = x + i\rho$, be a function which is periodic in u with a period of π and which, except for simple poles at $u = \xi_j + i\eta_j \cos\alpha$, $0 \leqq \xi_1 < \xi_2 < \cdots < \xi_n < \pi$, $\eta_j > 0$, is analytic in the strip S: $0 \leqq x \leqq \pi$. Further, let us denote the residue in $u = \xi_j + i\eta_j\cos\alpha$ by $a_j + b_j i \cos\alpha$ and assume that a_j and b_j are independent of α, and that $f(u,\alpha) \rightarrow c \pm di\cos\alpha$ as $\rho \rightarrow \pm\infty$ uniformly with respect to both x and α. Then, as is well known, we may write [5]

$$f(u,\alpha) = c + \sum_{j=1}^{\infty} (a_j + ib_j\cos\alpha)\cot(u - \xi_j - i\eta_j\cos\alpha).$$

In the u-plane, from which has been cut the rays $x = \xi_j + k\pi$, $|\rho| \geqq \eta_j$, $j = 1, 2, \cdots, n$, $k = 0, \pm 1, \pm 2, \cdots$, the function

$$\phi(u) = c + \sum_{j=1}^{\infty} (a_j\boldsymbol{i}_x + b_j\boldsymbol{i}_\rho)\cdot\boldsymbol{A}_\zeta \cot(u - \zeta_j)$$

will be single-valued, analytic and periodic with a period of π and likewise, the harmonic vector $\boldsymbol{H}(x,\rho) = (2\pi)^{-1}\int_0^{2\pi}\phi(u)\nabla u\,dt$ will be single-valued and periodic with a period of π.

The flow represented by this $\boldsymbol{H}(x,\rho)$ will behave, in the neighborhood of each circle $x = \xi_j + k\pi$, $\rho = \eta_j$, $k = 0, \pm 1, \pm 2, \cdots$, like the flow due to the superposition, on the circle, of a circular source filament of total strength a_j and a circular vortex filament of strength b_j. We thus obtain a result similar to the well-known "Kármán Street" of vortices.[6]

7. Energy in an axisymmetric force field.[7] Let

$$\boldsymbol{F}(x,\rho) = X(x,\rho)\boldsymbol{i}_x - {}^{-1}P(x,\rho)\boldsymbol{i}_\rho$$

represent the force in an axisymmetric field and let $f(x+i\rho)$ be the associate

[5] Cf. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, American Edition, p. 399, prob. 1.

[6] See L. M. Milne-Thompson, *Theoretical Hydrodynamics*, Macmillan (1938), p. 341.

[7] Cf. S. Bergman, *Bulletin of the American Mathematical Society*, vol. 49 (1943), pp. 163-174.

function. Then, using a somewhat more general formula than (2.3), we may write

$$\textbf{(7.1)} \qquad \boldsymbol{F}(x,\rho) = (2\pi i)^{-1} \int_{\Gamma} f(u)\, \nabla u \zeta^{-1} d\zeta,$$

where

$$\textbf{(7.2)} \qquad u = x + (i/2)\rho(\zeta + \zeta^{-1}), \qquad \nabla = \boldsymbol{i}_x(\partial/\partial x) + \boldsymbol{i}_\rho(\partial/\rho)$$

and Γ is a rectifiable closed curve in the ζ-plane.

Let us assume in what follows that $f(x + i\rho)$ is an algebraic function having the singularities $x + i\rho = u_1, u_2, \cdots, u_n$ in a ρ-convex region S of the u-plane. Let $\zeta_{jk}(x,\rho)$ be the roots of the equations (7.2) with $u = u_k$ and $k = 1, 2, \cdots, n$. Furthermore, let V be the locus of all points (x,ρ) for which at least two of the $\zeta_{jk}(x,\rho)$ are coincident ($j = 1, 2$; $k = 1, 2, \cdots, n$). The locus V will be called the "singular manifold of $f(u)$." Finally, let the region S' be a ρ-convex region drawn in the region obtained from S by omitting the points common to S and V.

If P_1: (x_1, ρ_1) is chosen as any point of S' and the Γ of formula (7.1) is taken as a curve Γ_1 enclosing one and only point $\zeta_{jk}(x_1,\rho_1)$, say $\zeta_{11}(x_1,\rho_1)$, formula (7.1) will yield a definite branch $\boldsymbol{F}_1(x,\rho)$ of the many-valued $\boldsymbol{F}(x,\rho)$. Now beginning at P_1, let us draw in S' a curve C_1 so short that, when (x,ρ) moves from P_1 along C_1, no point $\zeta_{jk}(x,\rho)$ crosses Γ_1. Hence, for all points (x,ρ) on C_1, formula (7.1) with $\Gamma = \Gamma_1$ furnishes the same branch $\boldsymbol{F}_1(x,\rho)$.

The work W_1, done on a unit particle moving along C_1, in the field $\boldsymbol{F}_1(x,\rho)$ will then be

$$\begin{aligned}
\textbf{(7.3)} \qquad W_1 &= \int_{C_1} \boldsymbol{F}(x,\rho) \cdot (\boldsymbol{i}_x dx + \boldsymbol{i}_\rho d\rho) \\
&= (2\pi i)^{-1} \int_{C_1} (\boldsymbol{i}_x dx + \boldsymbol{i}_\rho d\rho) \int_{\Gamma_1} f(u)\, \nabla u \zeta^{-1} d\zeta \\
&= (2\pi i)^{-1} \int_{\Gamma_1} \zeta^{-1} d\zeta \int_{K_1(\zeta)} f(u)\, du,
\end{aligned}$$

where $K_1(\zeta)$ is the curve described by the point u when ζ is a fixed point on Γ_1 and the point (x,ρ) describes the curve C_1. The quantity

$$\textbf{(7.4)} \qquad w_1(\zeta) = \int_{K_1(\zeta)} f(u)\, du$$

represents the work in the two dimensional field $f(u)$. The desired work W_1 is evidently expressed by formula (7.3) as the average of $w_1(\zeta)$ as ζ varies on Γ_1. That is, W_1 is connected with $w_1(\zeta)$ by way of the same operator as

given in (7.1); namely, to obtain (7.3), we have only to replace the vector $f(u)\nabla u$ in (7.1) by the scalar $(2\pi i)^{-1}w_1(\zeta)$.

If, instead of the curve C_1, we draw from P_1 in the region S', a curve C of arbitrary length, it will be necessary to establish the following lemma before obtaining results similar to (7.3).

LEMMA. *If C is an arbitrary rectifiable curve drawn from P_1 in S' and if it is divided into a sufficiently large number ν of non-overlapping equal arcs $C_1, C_2, \cdots, C_\nu$, then circles $\Gamma_1, \Gamma_2, \cdots, \Gamma_\nu$ can be drawn in the ζ-plane such that formula (7.1) will furnish the same branch $\boldsymbol{F}_1(x,\rho)$ for all points (x,ρ) on C, provided Γ in (7.1) is taken as Γ_k when the point (x,ρ) is on arc C_k.*

To prove this lemma, let us represent C parametrically in terms of its arc length s; i.e., C: $\omega(s) = x(s)\boldsymbol{i}_x + \rho(s)\boldsymbol{i}_\rho$, $0 \leqq s \leqq l$. Drawn in S', C does not intersect V and thus, for (x,ρ) on C, the points $\zeta_{jk}(s) = \zeta_{jk}[x(s), \rho(s)]$ are distinct. The distance between pairs of points $\zeta_{jk}(s)$ for each s will have a minimum $d > 0$, where d is independent of s. As the $\zeta_{jk}(s)$ are uniformly continuous on C, a $\delta > 0$ exists so that

$$|\zeta_{jk}(s') - \zeta_{jk}(s)| < d/2$$

for all j and k and all s and s' with $|s' - s| \leqq \delta$. Hence, if we let $s_1 = 0$, $s_2 = l/\nu, \cdots, s_\nu = (\nu - 1)l/\nu$, where $\nu > l/\delta$, and if we denote by C_p the arc $s_p \leqq s \leqq s_{p+1}$, then the circle Γ_p, where the equation of Γ_p is

$$|\zeta - \zeta_{11}(s_p)| = d/2, \tag{7.5}$$

will contain no singular point $\zeta_{jk}(s)$ other than $\zeta_{11}(s)$ for $s_p \leqq s \leqq s_{p+1}$. Therefore, if Γ in formula (7.1) is chosen as Γ_p when (x,ρ) is on C_p, formula (7.1) will always provide the same branch $F_1(x,\rho)$.

This lemma enables us to divide C up into a finite number of arcs drawn in S' and to derive for each a result similar to that given by formula (7.3). We may thus state the following theorem.

THEOREM V. *Let C be an arbitrary rectifiable curve in S', P_1: (x_1, ρ_1) any point on C and $\boldsymbol{F}_1(x,\rho)$ the branch of $\boldsymbol{F}(x,\rho)$ furnished by formula (7.1) when Γ encloses one and only singularity $\zeta_{jk}(x,\rho)$.*

Let $K(\zeta)$ be the locus of the point $u = x + (i/2)\rho(\zeta + \zeta^{-1})$ when ζ is fixed and the point (x,ρ) describes C. Let

$$w(\zeta) = \int_{K_1(\zeta)} f(u)\,du \tag{7.6}$$

be the work required to move a unit particle in the two-dimensional field $f(u)$ along the curve $K(\zeta)$.

Then ν circles $\Gamma_1, \Gamma_2, \cdots, \Gamma_\nu$ can be drawn in the ζ-plane such that the work

$$W = \int_C \boldsymbol{F}_1(x, \rho) \cdot (\boldsymbol{i}_x dx + \boldsymbol{i}_\rho d\rho) \tag{7.7}$$

required to move a particle in the three-dimensional axisymmetric field $\boldsymbol{F}_1(x, \rho)$ is the sum $W = W_1 + W_2 + \cdots + W_n$ of the transforms of $w(\zeta)$:

$$W_k = (2\pi i)^{-1} \int_{\Gamma_k} w(\zeta)\zeta^{-1} d\zeta.$$

A simple application of the above theorem is to the case where the force field $\boldsymbol{F}_1(x, \rho)$ is conservative. We may then express $\boldsymbol{F}(x, \rho)$ as the gradient of a potential $\Phi(x, \rho)$ thus: $\boldsymbol{F} = -\nabla\Phi$, $\nabla = \boldsymbol{i}_x\, \partial/\partial x + \boldsymbol{i}_\rho\, \partial/\partial\rho$, where

$$\Phi(x, \rho) = (2\pi i)^{-1} \int_\Gamma \phi(u)\zeta^{-1} d\zeta.$$

Thus, $f(u) = -\phi'(u)$. As the force field $f(u)$ is thus also conservative, we have for the work $w(\zeta)$ corresponding to the arc C_k:

$$w_k(\zeta) = \int_{u_{k+1}}^{u_k} \phi'(u)\, du = \phi(u_k) - \phi(u_{k+1})$$

where $u_k = x_k + (i/2)\rho_k(\zeta + \zeta^{-1})$ and where (x_k, ρ_k) and (x_{k+1}, ρ_{k+1}) are the end-points of C_k. As the Γ_k have been determined so that (7.1) will always furnish the same branch $\boldsymbol{F}_1(x, \rho)$ of the force function, the sum W of the W_k is

$$W = \sum_{k=1}^{\nu} W_k = \Phi(x_1, \rho_1) - \Phi(x_{\nu+1}, \rho_{\nu+1}),$$

a result to be expected in a conservative force field.

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AN EXTENSION OF A PERRON SYSTEM OF LINEAR EQUATIONS IN INFINITELY MANY UNKNOWNS.*

By I. M. Sheffer.

1. Introduction. Consider the system of equations

$$(1.1) \qquad \sum_{s=0}^{\infty} (a_s + b_{n,n+s}) x_{n+s} = c_n, \qquad (n = 0, 1, \cdots).$$

Let the following conditions [(1.2) to (1.5)] be called *conditions* P_1:

$$(1.2) \qquad F(t) \equiv \sum_{s=0}^{\infty} a_s t^s$$

is analytic in $|t| \leqq 1$;

$$(1.3) \qquad |b_{n,n+s}| \leqq k_n \theta^s, \qquad \theta < 1;$$

$$(1.4) \qquad k_n \to 0;$$

$$(1.5) \qquad a_0 + b_{nn} \neq 0, \qquad (n = 0, 1, \cdots).$$

For this system Perron [1] has proved the following theorem:

Let conditions P hold. If $F(t)$ has λ zeros in $|t| \leqq 1$, then for every $\{c_n\}$ for which $\limsup |c_n|^{1/n} \leqq 1$ the general solution $\{x_n\}$ of (1.1) with $\limsup |x_n|^{1/n} \leqq 1$ contains λ arbitrary constants. There exist λ linearly independent solutions $\{x_n^{(i)}\}$, $(i = 1, \cdots, \lambda)$, of the homogeneous system with $\limsup |x_n^{(i)}|^{1/n} \leqq 1$, and if $\{x_n^{(0)}\}$ is a solution of (1.1) then the general solution (with $\limsup |x_n|^{1/n} \leqq 1$) can be written

$$(1.6) \qquad x_n = x_n^{(0)} + \sum_{i=1}^{\lambda} \alpha_i x_n^{(i)}, \qquad (n = 0, 1, \cdots),$$

where $\alpha_1, \cdots, \alpha_\lambda$ are arbitrary. Moreover, an integer M exists, independent of the sequence $\{c_n\}$, with the property that (1.1) has one and only one solution $\{x_n\}$ ($\limsup |x_n|^{1/n} \leqq 1$) for which $x_M, x_{M+1}, \cdots, x_{M+\lambda-1}$ are preassigned.

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* Received January 3, 1944.

[1] O. Perron, "Über Summengleichungen und Poincarésche Differenzengleichungen," *Mathematische Annalen*, vol. 84 (1921), pp. 1-15.

It is our purpose here to enlarge the Perron system. We shall consider the system of equations

$$(1.7)\qquad \sum_{s=0}^{n-1} b_{ns}x_s + \sum_{s=0}^{\infty}(a_s + b_{n,n+s})x_{n+s} = c_n, \qquad (n = 0, 1, \cdots).$$

As the *limit superior* of a sequence occurs often in the discussion, we introduce the following name and notation.

Definition. By the *type* of a sequence $\{x_n\}$ is meant the number $((x_n))$ given by

$$(1.8)\qquad ((x_n)) \equiv \limsup |\, x_n \,|^{1/n}.$$

We impose on the system (1.7) the following conditions [(i) and (ii)] to be referred to as *conditions Q*:

(i) Conditions P hold (save that (1.5) is no longer assumed).

(ii) There exists a number ρ in $0 \leqq \rho < 1$ such that

$$(1.9)\qquad \Big(\Big(\sum_{s=0}^{n-1} b_{ns}x_s\Big)\Big) \leqq \rho$$

for every $\{x_n\}$ for which $((x_n)) \leqq 1$. Condition (ii) is completely equivalent to the following: If

$$(1.10)\qquad C^*_n(t) \equiv \sum_{s=0}^{n-1} |\, b_{ns} \,|\, t^s, \qquad (n = 0, 1, \cdots),$$

then given $\epsilon > 0$, there corresponds a $\delta_\epsilon > 0$ such that

$$(1.11)\qquad ((C^*_n(t))) < \rho + \epsilon$$

for all t in $|\, t \,| \leqq 1 + \delta_\epsilon$. (Or, $((C^*_n(1 + \delta_\epsilon))) < \rho + \epsilon$.)

Observe that the system (1.1) is a diagonal system; in other words $x_0, \cdots, x_{n-1}$ occur only in the first n equations. The system (1.7) does not enjoy this property, and therein lies the source of whatever difference there is in the two systems, as we shall see.

The genesis of the Perron system is the linear recurrence relation

$$(1.12)\qquad p_0x_n + p_1x_{n+1} + \cdots + p_\lambda x_{n+\lambda} = c_n, \qquad (n = 0, 1, \cdots),$$

with $p_0p_\lambda \neq 0$. The corresponding homogeneous system has exactly λ linearly independent solutions, and that these are all of finite type is seen by exhibiting a fundamental set of solutions:

$$x_n = t_i^n, nt_i^n, \cdots, n^{k_i-1}t_i^n, \qquad (i = 1, \cdots, \sigma), \tag{1.13}$$

where $t_1, \cdots, t_\sigma$ are the distinct zeros of the polynomial

$$P(t) = p_0 + p_1 t + \cdots + p_\lambda t^\lambda, \tag{1.14}$$

and $k_1, \cdots, k_\sigma$ are the corresponding orders, so that $k_1 + \cdots + k_\sigma = \lambda$. The system (1.12) itself has as a particular solution, corresponding to every sequence $\{c_n\}$ of finite type (say $((c_n)) = R$),

$$x_n = \frac{1}{2\pi i}\int_\Gamma C(1/t)\,\frac{t^{n-1}}{P(t)}\,dt, \tag{1.15}$$

where $C(t) = \sum_0^\infty c_n t^n$ is analytic in $|t| < 1/R$ and where Γ is the circle $|t| = R'$, arbitrary save for the two restrictions:

$$R' \neq |t_i|, \ (i = 1, \cdots, \sigma); \qquad R' > R.$$

From (1.15) we see that $((x_n)) \leqq R'$, and from the arbitrariness of R' $(> R)$ we conclude that (1.12) has a solution of type not exceeding R. Actually, this solution is of type equal to R, for if it were of lesser type. then the left side of (1.12) would be of lesser type; i.e., $((c_n)) < R$, which is a contradiction.

An extension of (1.12), in the homogeneous case, was made by Poincaré,[2] who investigated the asymptotic character of solutions of the system

$$a_{n0}x_n + a_{n1}x_{n+1} + \cdots + a_{n\lambda}x_{n+\lambda} = 0, \qquad (n = 0, 1, \cdots), \tag{1.16}$$

where $a_{ni} \to p_i$, $(i = 0, \cdots, \lambda)$.

The system (1.16) may be regarded as the result of adding a perturbation element to (1.12) (homogeneous case), where the perturbation is small enough to preserve certain properties of the solutions of the original system. Perron introduced his system (1.1), in part, for the purpose of sharpening the conclusions of Poincaré. The system (1.16) is a particular case of (1.1), and we may think of (1.1) as representing the addition of a new perturbation element to (1.12). And in this sense, (1.7) provides still another perturbation.

Concerning the Perron theorem some words are in order as to the *conditions P*. Suppose that $F(t)$ is analytic in $|t| \leqq R$. Then $\sum_{s=0}^\infty a_s x_{n+s}$ con-

[2] H. Poincaré, "Sur les Equations Linéaires aux Différentielles ordinaires et aux Différences finies," *American Journal of Mathematics*, vol. 7 (1885), pp. 203-258.

verges for every $\{x_n\}$ for which $((x_n)) \leqq R$. The type of $\{x_n\}$ may even be larger, how large depending on how much greater than R is the radius of convergence of (1.2). Since, however, we wish to have a theorem applicable to *every* $F(t)$ analytic in $|t| \leqq R$, we must insist that $((x_n)) \leqq R$. In this case, the sequence $\sum_{s=0}^{\infty} a_s x_{n+s}$ is also of type not exceeding R. Again, to have $y_n \equiv \sum_{s=0}^{\infty} b_{n,n+s} x_{n+s}$ exist for all such $\{x_n\}$, we must assume, *if we insist on a condition of the form* (1.3), (1.4), that $\theta < 1/R$; and then it follows that $((y_n)) \leqq R$. The left side of (1.1) is then a sequence of type not exceeding R, and so we naturally assume the condition $((c_n)) \leqq R$ for $\{c_n\}$.

If we now make the substitutions

$$c'_n = c'/R^n, \quad x'_n = x_n/R^n, \quad a'_s = R^s a_s, \quad b'_{n,n+s} = R^s b_{n,n+s},$$

then (1.1) assumes the same form with all letters primed; and for this new system the Perron conditions hold. This shows that the conditions as to type are not restrictive. On the other hand, the *nature* of conditions (1.3), (1.4) is a restriction, and it is because of this that the perturbed system has properties analogous to those of the simple system (1.12).

With these preliminary remarks we turn to the system (1.7). We shall apply the method of Perron wherever possible.

2. The $\mathfrak{D}$-system of equations. We shall refer to (1.7) as the $\mathfrak{D}$ system:

$$(2.1) \qquad \mathfrak{D}: \quad \sum_{s=0}^{n-1} b_{ns} x_s + \sum_{s=0}^{\infty} (a_s + b_{n,n+s}) x_{n+s} = c_n, \qquad (n = 0, 1, \cdots);$$

and for $\mathfrak{D}$ we assume conditions Q with the further condition $F(t) \not\equiv 0$.

Let $\epsilon > 0$ be given such that $\rho + \epsilon < 1$. Choose $\delta_\epsilon > 0$ to satisfy (1.11), and δ'_ϵ in $0 < \delta'_\epsilon < \delta_\epsilon$ so that

$$(2.2) \qquad ((C^*_n(t))) < \rho + \epsilon/2$$

for all t in $|t| \leqq |1 + \delta'_\epsilon$. A number R can be found to satisfy the following conditions [(i) to (iii)]:

(i) $R\theta < 1 < R \leqq 1 + \delta'_\epsilon$.

(ii) $F(t)$ and $B_n(t)$ are analytic in $|t| \leqq R$, where

$$(2.3) \qquad B_n(t) = \sum_{s=0}^{\infty} b_{n,n+s} t^{n+s}, \qquad (n = 0, 1, \cdots).$$

(iii) $F(t) \neq 0, \qquad 1 < |t| \leqq R.$

Define $A_n(t)$, $C_n(t)$, $D_n(t)$ by the series

$$A_n(t) = \sum_{s=0}^{\infty} a_s t^{n+s} = t^n F(t), \qquad (n = 0, 1, \cdots), \tag{2.4}$$

$$C_n(t) = \sum_{t=1}^{n-1} b_{n_1 s} t^s, \qquad (n = 0, 1, \cdots), \tag{2.5}$$

$$D_n(t) = A_n(t) + B_n(t) + C_n(t), \qquad (n = 0, 1, \cdots). \tag{2.6}$$

Let Γ_r be the circle $|t| = r$, where r is arbitrary in $1 < r \leqq R$. On Γ_r we have, uniformly,

$$| C_n(t) | \leqq \alpha_r(\rho + \epsilon)^n, \qquad | B_n(t) | \leqq k_n r^n/(1 - r\theta). \tag{2.7}$$

Let $((x_n)) \leqq 1$, so that the series

$$X(t) = \sum_{s=0}^{\infty} x_s t^s \tag{2.8}$$

has a radius of convergence at least as great as one. If r lies in $1 < r < R$, then

$$x_n = \frac{1}{2\pi i} \int_{\Gamma_r} X(1/t) t^{n-1} dt, \qquad (n = 0, 1, \cdots). \tag{2.9}$$

The system (2.1) can then be written

$$c_n = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{D_n(t)}{t} X(1/t) dt = \frac{1}{2\pi i} \int (1/t) X(1/t) \{C_n(t) + t^n F(t) + B_n(t)\} dt, \qquad (n = 0, 1, \cdots). \tag{2.10}$$

Uniformity of convergence permits term-by-term integration. Conversely, from (2.10) we get (2.1) so that we have

LEMMA 1. *Let conditions Q hold, with $((c_n)) \leqq 1$. Every sequence $\{x_n\}$, with $((x_n)) \leqq 1$, that satisfies* (2.1) *also satisfies* (2.10), *and conversely.*

We now apply a transformation, utilized by Perron, that will replace $F(t)$ by a polynomial.

LEMMA 2. *If $((u_n)) \leqq a$, and if $G(t) = \sum_{s=0}^{\infty} g_s t^s$ is analytic in $|t| < g$ where $g > a$, then*

$$v_n = \sum_{s=0}^{\infty} g_s u_{n+s} \tag{2.11}$$

is a sequence of type not exceeding a.

To show this, let σ be chosen in $1/g < \sigma < 1/a$, and let Γ_σ be the circle $|t| = \sigma$. Then

$$v_n = \frac{1}{2\pi i}\int_{\Gamma_\sigma}(1/t)G(1/t)\{u_n + u_{n+1}t + \cdots\}dt.$$

Since $|u_n| \leqq A_j \cdot \gamma^n$ for every γ in $a < \gamma < 1/\sigma$, we see that $u_n + u_{n+1}t + \cdots \ll A_j\gamma^n/(1-\gamma t)$ on Γ_σ, so that $((v_n)) \leqq \gamma$; and from the arbitrariness of γ (in the range $\alpha < \gamma < 1/\sigma$) we conclude that $((v_n)) \leqq a$.

Let $F(t)$ have λ zeros in $|t| \leqq 1$, say $t = t_1, \cdots, t_\sigma$, of respective orders $k_1, \cdots, k_\sigma$ so that $k_1 + \cdots + k_\sigma = \lambda$. On defining $P(t)$ by

$$P(t) = \prod_{i=1}^{\sigma}(t - t_i)^{k_i} \equiv p_0 + p_1 t + \cdots + p_\lambda t^\lambda \qquad (p_\lambda = 1) \tag{2.12}$$

we have

$$F(t) = P(t)J(t) \tag{2.13}$$

where $J(t)$ is analytic and different from zero in $|t| \leqq R$. (If $\lambda = 0$ then $P(t) \equiv 1$, $J(t) \equiv F(t)$.)

The function $1/J(t)$ is also analytic in $|t| \leqq R$, since $F(t) \neq 0$ in $1 < |t| \leqq R$, so that if

$$1/J(t) = \sum_{n=0}^{\infty} h_n t^n, \tag{2.14}$$

then $((h_n)) < 1/R$.

In Lemma 2 we may make the choice $G(t) = 1/J(t)$, $\{u_n\} = \{c_n\}$. Then (2.10) becomes

$$\gamma_n = \frac{1}{2\pi i}\int_{\Gamma_r}(1/t)X(1/t)\{t^n P(t) + \sum_{s=0}^{\infty} h_s[B_{n+s}(t) + C_{n+s}(t)]\}dt, \qquad (n = 0, 1, \cdots),$$

where

$$\gamma_n = h_0 c_n + h_1 c_{n+1} + \cdots, \qquad (n = 0, 1, \cdots). \tag{2.15}$$

Absolute and uniform convergence on Γ_r is guaranteed by (2.7); and $((\gamma_n)) \leqq 1$ by Lemma 2.

Define new functions $C_n^\dagger(t)$, $B_n^\dagger(t)$ by

$$C_n^\dagger(t) = \sum_{j=0}^{n-1}\left(\sum_{s=0}^{\infty} h_s b_{n+s,j}\right)t^j, \tag{2.16}$$

$$B_n^\dagger(t) = \sum_{j=n}^{\infty}\left(\sum_{s=0}^{\infty} h_s b_{n+s,j}\right)t^j. \tag{2.17}$$

Each double series converges absolutely and uniformly on Γ_r, and we have the system

$$(2.18)\quad \gamma_n = \frac{1}{2\pi i}\int_{\Gamma_r} (1/t)X(1/t)\{C_n{}^\dagger(t) + t^nP(t) + B_n{}^\dagger(t)\}dt, \qquad (n=0,1,\cdots),$$

which, when expanded, becomes the system $\mathfrak{D}^\dagger$:

$$(2.19)\quad \mathfrak{D}^\dagger:\ \sum_{s=0}^{n-1}\beta_{ns}x_s + \sum_{s=0}^{\lambda}(p_s+\beta_{n,n+s})x_{n+s} + \sum_{s=\lambda+1}^{\infty}\beta_{n,n+s}x_{n+s} = \gamma_n, \qquad (n=0,1,\cdots),$$

where

$$(2.20)\qquad \beta_{nj} = \sum_{s=0}^{\infty} h_s b_{n+s,j}, \qquad (n,j=0,1,\cdots).$$

$\mathfrak{D}^\dagger$ is thus a consequence of $\mathfrak{D}$.

The system $\mathfrak{D}^\dagger$ satisfies a set of conditions Q. To see this, choose ω such that $1+\delta'_\epsilon < \omega^{-1} \leqq 1+\delta_\epsilon$ and let Γ_ω be the circle $|t| = \omega$. For $j < n+s$,

$$(2.21)\quad |b_{n+s,j}| = \frac{1}{2\pi i}\int_{\Gamma\omega} C^*{}_{n+s}(1/t)t^{j-1}dt,$$

$$\sum_{s=0}^{\infty}|h_s b_{n+s,j}| = \frac{1}{2\pi i}\int_{\Gamma\omega} t^{j-1}\{\sum_{s=0}^{\infty}|h_s|C^*{}_{n+s}(1/t)\}dt \leqq \alpha_\omega A_\epsilon \omega^j(\rho+\epsilon)^n;$$

where $A_\epsilon = \sum_{s=0}^{\infty}|h_s|(\rho+\epsilon)^s$. Hence

$$(2.22)\qquad |\beta_{nj}| \leqq \alpha_\omega A_\epsilon \omega^j(\rho+\epsilon)^n, \qquad j<n;$$

so that for all t in $|t| \leqq 1+\delta'_\epsilon$ (and therefore for $|t| \leqq R$),

$$C_n{}^{\dagger *}(t) \equiv \sum_{j=0}^{n-1}|\beta_{nj}|t^j \ll \alpha_\omega A_\epsilon(\rho+\epsilon)^n/(1-\omega t).$$

Consequently,

$$(2.23)\qquad ((C_n{}^{\dagger *}(t))) \leqq \rho+\epsilon$$

for all t in $|t| \leqq 1+\delta'_\epsilon$. In view of the independence of $C_n{}^{\dagger *}(t)$ relative to ϵ, this is equivalent to

$$(2.24)\qquad ((\sum_{s=0}^{n-1}\beta_{ns}x_s)) \leqq \rho$$

for every $\{x_n\}$ for which $((x_n)) \leqq 1$. That is, the relation (ii) of conditions Q is fulfilled.

Now suppose $j \geqq n$. Then

$$\beta_{nj} = \sum_{s=0}^{j-n} h_s b_{n+s,j} + \sum_{s=j+1}^{\infty} h_{s-n}b_{sj} \equiv I_1 + I_2.$$

we have

$$| I_1 | \leqq \sum_{s=0}^{j-n} | h_s | k_{n+s}\theta^{j-n-s},$$

so that, if θ'' is chosen in $\theta < 1/R < \theta'' < 1$, and if

$$l_n = \max\{k_n, k_{n+1}, \cdots\} \qquad (l_n \to 0),$$

then

$$| I_1 | \leqq Bl_n\theta''^{j-n},$$

where $B = \sum_{s=0}^{\infty} | h_s | \theta''^{-s}$. Again,

$$| I_2 | \leqq \sum_{s=j+1}^{\infty} | h_{s-n}b_{sj} |,$$

so that from (2.21), which applies since $j < s$ in b_{sj},

$$| I_2 | \leqq \alpha_\omega \cdot \omega^j A_\epsilon(\rho + \epsilon)^n = \alpha_\omega A_\epsilon[\omega(\rho + \epsilon)]^n \cdot \omega^{j-n}.$$

Since $\omega < 1$, $\omega(\rho + \epsilon) < 1$, it is possible to choose θ' to lie in $\max\{\theta'', \omega\} \leqq \theta' < 1$; and on defining k'_n by

$$k' = Bl_n + \alpha_\omega A_\epsilon[\omega(\rho + \epsilon)]^n,$$

then

$$| \beta_{nj} | \leqq k'_n\theta'^{j-n}, \qquad j \geqq n.$$

That is,

$$(2.25) \qquad | \beta_{n,n+s} | \leqq k'_n\theta'^s,$$

with $0 < \theta' < 1$ and $k'_n \to 0$.

In other words, the system $\mathfrak{D}^\dagger$ satisfies a set of conditions Q, with θ replaced by a number θ' having the same property. Any larger number will serve equally well if it is less than one.

The system (2.19) is a consequence of (2.1). But also, starting from (2.19), and using $G(t) = J(t)$, $\{u_n\} = \{\gamma_n\}$ in Lemma 2, it is a simple matter to demonstrate that the system (2.1) results. We accordingly have

LEMMA 3. *Let the system* (2.1) *fulfill conditions Q, and let* $((c_n)) \leqq 1$. *Then the system* (2.19) *also has the property Q (with θ' replacing θ), and* $((\gamma_n)) \leqq 1$. *Moreover, every sequence $\{x_n\}$, of type not exceeding one, that satisfies* (2.1) *also satisfies* (2.19) *and conversely.*

We now consider a further reduction [Perron, *loc. cit.*].

Definition. Given a system of linear forms (or equations) $\mathfrak{F}$: $\sum_{j=0}^{\infty} f_{ij}x_j$,

$(j=0,1,\cdots)$. By the *truncate of* $\mathfrak{F}$ *of order* m, denoted by $\mathfrak{F}^{(m)}$, is meant the system obtained from $\mathfrak{F}$ by dropping the first m forms (or equations). In particular, $\mathfrak{F}$ is its own truncate of order zero.

The transformation $\{u_n\}$ to $\{v_n\}$ of Lemma 2 was applied to carry the system $\mathfrak{D}$ into $\mathfrak{D}^\dagger$. Since γ_n does not depend on $c_0,\cdots,c_{n-1}$, we conclude that *the same transformation of* $\mathfrak{D}$ *to* $\mathfrak{D}^\dagger$ *carries every truncate of* $\mathfrak{D}$ *into the truncate of* $\mathfrak{D}^\dagger$ *of the same order.*

THEOREM 1. *Let the system* $\mathfrak{D}$ *satisfy conditions* Q, *with* $((c_n))\leqq 1$. *There exists a number* ζ *in* $1<\zeta$, *and an integer* $M\geqq 0$, *both independent of* $\{c_n\}$, *such that the* M*-th truncate system* $\mathfrak{D}^{(M)}$ *has one and only one solution* $\{x_n\}$ *of type not exceeding* ζ, *for which the values* $x_0,\cdots,x_{\lambda+M-1}$ *are assigned. Moreover, this solution is of type not exceeding one.*

The proof is divided into two parts: uniqueness and existence. (Compare Perron, *loc. cit.*) First we take up uniqueness. Instead of the system $\mathfrak{D}$ we may use the equivalent system $\mathfrak{D}^\dagger$.

Let the non-negative integer M be arbitrary at present. Define $\{\delta_n\}$ by

$$(2.26)\qquad \sum_{s=0}^{\infty}(\delta_s/t^s)\equiv t^\lambda/P(t)=\prod_{i=1}^{\lambda-q}(1-r_i/t)^{-1},$$

where we express $P(t)$ in the form

$$P(t)=t^q(t-r_1)\cdots(t-r_{\lambda-q}),\qquad 0<|r_i|\leqq 1,$$

and let

$$(2.27)\qquad c'_n=\delta_{n-M}\gamma_M+\delta_{n-M-1}\gamma_{M+1}+\cdots+\delta_0\gamma_n,\qquad n\geqq M.$$

From (2.26) we obtain the relations

$$(2.28)\qquad \begin{aligned}&\delta_0p_{\lambda-i}+\delta_1p_{\lambda-i+1}+\cdots+\delta_ip_\lambda=\begin{cases}1, & i=0\\ 0, & i=1,\cdots,\lambda;\end{cases}\\ &\delta_np_0+\delta_{n+1}p_1+\cdots+\delta_{n+\lambda}p_\lambda=0, \qquad\qquad (n=0,1,\cdots);\end{aligned}$$

so that the system

$$(2.29)\qquad x_{\lambda+n}+\sum_{s=0}^{\lambda-1}\sum_{r=0}^{s}p_{s-r}\delta_{n-M-r}x_{M+s}=c'_n-\sum_{r=M}^{n}\sum_{s=0}^{\infty}\delta_{n-r}\beta_{rs}x_s\qquad (n\geqq M)$$

follows from $\mathfrak{D}^\dagger$ and (2.27).

The series on the right side converge for all $\{x_n\}$ of type not exceeding R since each equation of (2.29) is only a linear combination of a finite number of equations of $\mathfrak{D}^\dagger$. Moreover, we can go from (2.29) back to the M-th order truncate of $\mathfrak{D}^\dagger$ by inversion of these linear combinations in an obvious manner.

The number δ'_ϵ has already been defined. By increasing θ', if necessary, we may assume that it lies in the range $1 < 1/\theta' < 1 + \delta'_\epsilon$. Now choose ζ, ζ_1, ζ_2 to satisfy

$$(2.30) \qquad 1 < \zeta_1 < \zeta < \zeta_2 < 1/\theta' < 1 + \delta'_\epsilon.$$

The following inequality is obvious:

$$\binom{\lambda + n - r - 1}{\lambda - 1} \leqq (\lambda + n)^{\lambda - 1} \leqq (\lambda + n)^\lambda, \qquad 0 \leqq r \leqq n.$$

Since λ is fixed, the right hand member is a sequence of type one, so that an integer M_1 can be found such that

$$(2.31) \qquad \frac{\alpha_\omega A_\epsilon}{(1 - \omega\zeta_2)(1 - \rho - \epsilon)} (\lambda + n)^\lambda < \tfrac{1}{4}\zeta_2^{\lambda + n}, \qquad n \geqq M_1.$$

Further, indices M_2, M_3, M_4 exist so that

$$(2.32) \qquad \frac{\alpha_\omega A_\epsilon (\lambda + n)^\lambda}{(1 - \omega\zeta_1)(1 - \rho - \epsilon)} < \tfrac{1}{4}\zeta_1^{\lambda + n}, \qquad n \geqq M_2;$$

$$(2.33) \qquad k'_n < \tfrac{1}{2}(1 - \theta'\zeta_2)(\zeta_2 - 1)^\lambda, \qquad n \geqq M_3;$$

$$(2.34) \qquad k'_n < \tfrac{1}{4}(\zeta_1 - 1)^\lambda (1 - \theta'\zeta_1), \qquad n \geqq M_4.$$

Now choose M as any integer (once chosen, fixed) exceeding $\max\{M_1, M_2, M_3, M_4\}$. Suppose that $\{x_n\}$, $\{y_n\}$ are two sequences of type not exceeding ζ, that each satisfies the truncate system $\mathfrak{D}^{\dagger(M)}$, and that $x_n = y_n$, $0 \leqq n \leqq \lambda + M - 1$. We shall show that $x_n \equiv y_n$. Let $z_n = x_n - y_n$. Then from (2.29) we see that $\{z_n\}$ satisfies the system

$$(2.35) \qquad z_{\lambda+n} = -\sum_{r=M}^{n} \sum_{s=\lambda+M}^{\infty} \delta_{n-r}\beta_{rs}z_s, \qquad n \geqq M.$$

We can write

$$z_{\lambda+n} \equiv Q_1 + Q_2$$

where

$$Q_1 = -\sum_{r=M}^{n} \sum_{s=\lambda+M}^{r-1} \delta_{n-r}\beta_{rs}z_s, \qquad Q_2 = -\sum_{r=M}^{n} \sum_{s=r}^{\infty} \delta_{n-r}\beta_{rs}z_s.$$

Now $((z_n)) \leqq \zeta$, so that a *minimum constant* $C \geqq 0$ exists for which

$$(2.36) \qquad |z_n| \leqq C\zeta_2{}^n, \qquad n \geqq 0;$$

and since $z_n = 0$ for $n < \lambda + M$, the equality in (2.36) is achieved for some value $n \geqq \lambda + M$. Then

$$| Q_1 | \leqq C \sum_{r=M}^{n} | \delta_{n-r} | C_r{}^{\dagger *}(\zeta_2) \leqq \frac{C\alpha_\omega A_\epsilon}{1-\omega\zeta_2} \sum_{r=M}^{n} | \delta_{n-r} | (\rho+\epsilon)^r,$$

where $C_r{}^{\dagger *}$ is defined just before (2.23). From (2.26),

$$\sum_{s=0}^{\infty} (\delta_s/t^s) << \prod_{t=1}^{\lambda} (1-1/t)^{-1} = \sum_{s=0}^{\infty} \binom{\lambda+s-1}{s} (1/t^s),$$

so that [Perron, *loc. cit.*]

(2.37) $$| \delta_s | \leqq \binom{\lambda+s-1}{s} = \binom{\lambda+s-1}{\lambda-1}.$$

Hence

$$| Q_1 | \leqq \frac{C\alpha_\omega A_\epsilon}{1-\omega f_2} (\lambda+n)^\lambda \cdot \frac{1}{1-\rho-\epsilon} < \tfrac{1}{4} C\zeta_2{}^{\lambda+n}.$$

Again, from (2.25),

$$| Q_2 | \leqq C \sum_{r=M}^{n} \binom{\lambda+n-r-1}{n-r} k'_r \sum_{s=r}^{\infty} \theta'^{s-r}\zeta_2{}^s$$
$$\leqq \tfrac{1}{2} C(\zeta_2-1)^\lambda \sum_{r=M}^{n} \binom{\lambda+n-r-1}{n-r} \zeta_2{}^r \leqq \tfrac{1}{2} C(\zeta_2-1)^\lambda \cdot \zeta_2{}^n \sum_{q=0}^{\infty}$$
$$\binom{\lambda+q-1}{q} \zeta_2{}^{-q} = (C/2)\zeta_2{}^{\lambda+n}.$$

This gives the relation

$$| z_{\lambda+n} | \leqq \tfrac{3}{4} C\zeta_2{}^{\lambda+n}, \qquad n \geqq M.$$

This is inconsistent with (2.36), since C is minimal, unless $C = 0$. Consequently $z_n = 0$ for all n, and the uniqueness is hereby established.

We turn to the problem of existence. Let

(2.38) $$x_i = X_i, \qquad (i = 0, 1, \cdots, \lambda+M-1)$$

be assigned. Define $x_i{}^{(0)}$ by

(2.39) $$x_i{}^{(0)} = X_i, \quad i < \lambda+M; \qquad x_i{}^{(0)} = 0, \quad i \geqq \lambda+M.$$

Then define a sequence of approximations $x_i{}^{(k)}$, $(k = 1, 2, \cdots)$ by

(2.40) $$x_i{}^{(k)} = x_i{}^{(k-1)}, \qquad i < \lambda+M;$$

$$x_{\lambda+n}{}^{(k)} + \sum_{s=0}^{\lambda-1} \sum_{r=0}^{s} p_{s-r}\delta_{n-M-r}x_{M+s}{}^{(k-1)} = c'_n - \sum_{r=M}^{n} \sum_{s=0}^{\infty} \delta_{n-r}\beta_{rs}x_s{}^{(k-1)}, \quad n \geqq M.$$

For $k > 1$ we obtain the inequality

(2.41) $$| x_{\lambda+n}{}^{(k)} - x_{\lambda+n}{}^{(k-1)} | \leqq \sum_{r=M}^{n} \sum_{s=\lambda+M}^{\infty} \binom{\lambda+n-r-1}{n-r} | \beta_{rs} | \; | x_s{}^{(k-1)} - x_s{}^{(k-2)} |,$$

and for $k=1$:

$$(2.42)\quad | x_{\lambda+n}{}^{(1)} - x_{\lambda+n}{}^{(0)} | \leqq | c'_n | + \sum_{s=0}^{\lambda-1} \sum_{r=0}^{s} \binom{\lambda+n-M-r-1}{n-M-r} | p_{s-r} | \, | X_{M \cdot} \,\, + \sum_{r=M}^{n} \sum_{s=0}^{\lambda+M-1} \binom{\lambda+n-r-1}{n-r} | \beta_{rs} | \, | X_s | \equiv | c'_n | + U_1 + U_2.$$

Now $((\gamma_n)) \leqq 1$. Hence if ξ is chosen in $1 < \xi < \zeta_1$, we have $| \gamma_n | \leqq d\xi^n$ for a suitable d. From (2.27), then,

$$| c'_n | \leqq d\xi^n \sum_{s=0}^{n-M} \binom{\lambda+s-1}{s} \xi^{-s} \leqq d\xi^n \sum_{s=0}^{\infty} \binom{\lambda+s-1}{s} \xi^{-s};$$

so that a number Λ_1 exists such that

$$| c'_n | \leqq \Lambda_1 \zeta_1{}^{n+\lambda}, \qquad n \geqq M.$$

Let

$$\sigma_1 = \max\{| X_0 |, \cdots, | X_{\lambda+M-1} |\}, \qquad \sigma_2 = \max\{| p_0 |, \cdots, | p_\lambda |\}.$$

Then

$$U_1 \leqq \sigma_1\sigma_2 \sum_{s=0}^{\lambda-1} \sum_{r=0}^{s} \binom{\lambda+n-M-r-1}{\lambda-1} \leqq R_1(\lambda+n)^\lambda \leqq \Lambda_2 \zeta_1{}^{\lambda+n}$$

for a suitably chosen R_1 and Λ_2. Also,

$$U_2 \leqq \sigma_1 \sum_{r=M}^{n} \binom{\lambda+n-r-1}{n-r} \{ \sum_{s=0}^{r-1} | \beta_{rs} | + \sum_{s=r}^{\infty} | \beta_{rs} | \}$$
$$\leqq \sigma_1 \sum_{r=M}^{n} \binom{\lambda+n-r-1}{\lambda-1} \left\{ \frac{\alpha_\omega A_\epsilon}{1-\omega} (\rho+\epsilon)^r + k'_r/(1-\theta') \right\}.$$

The brace is bounded, since it approaches 0 as $r \to \infty$. Hence if σ_3 is an upper bound for it,

$$U_2 \leqq \sigma_1\sigma_3 \cdot n(\lambda+n)^{\lambda-1} \leqq \sigma_1\sigma_2(\lambda+n)^\lambda < \Lambda_3 \zeta_1{}^{\lambda+n}$$

for a suitable Λ_3.

If, then, $\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3$ we obtain the inequality

$$(2.43)\qquad | x_{\lambda+n}{}^{(1)} - x_{\lambda+n}{}^{(0)} | < \Lambda \zeta_1{}^{\lambda+n}, \qquad n \geqq M.$$

Using this relation,

$$| x_{\lambda+n}{}^{(2)} - x_{\lambda+n}{}^{(1)} | \leqq \Lambda \sum_{r=M}^{n} \binom{\lambda+n-r-1}{n-r} \{ \sum_{s=\lambda+M}^{r-1} | \beta_{rs} | \zeta_1{}^s + \sum_{s=r}^{\infty} | \beta_{rs} | \zeta_1{}^s \} \equiv \Lambda\{T_1 + T_2\}.$$

It is readily found that

$$T_1 \leqq \frac{\alpha_\omega A_\epsilon}{1-\omega\zeta} \sum_{r=M}^{n} \binom{\lambda+n-r-1}{\lambda-1} (\rho+\epsilon)^r \leqq \frac{\alpha_\omega A_\epsilon(\lambda+n)^n}{(1-\omega\zeta_1)(1-\rho-\epsilon)} < \tfrac{1}{4}\zeta_1{}^{\lambda+n};$$

and that

$$T_2 \leqq \sum_{r=M}^{n} \binom{\lambda+n-r-1}{n-r} \zeta_1{}^r \cdot k'_r \sum_{q=0}^{\infty} (\theta'\zeta_1)^q \leqq \tfrac{1}{4}(\zeta_1-1)^\lambda \sum_{r=M}^{n} \binom{\lambda+n-r-1}{n-r} \zeta_1{}^r$$

$$\leqq \tfrac{1}{4}(\zeta_1-1)^\lambda \cdot \zeta_1{}^n \sum_{q=0}^{\infty} \binom{\lambda+q-1}{q} \zeta_1{}^{-q} = \tfrac{1}{4}\zeta_1{}^{\lambda+n}.$$

Hence

$$| x_{\lambda+n}{}^{(2)} - x_{\lambda+n}{}^{(1)} | \leqq \tfrac{1}{2}\Lambda\zeta_1{}^{\lambda+n}, \qquad n \geqq M.$$

The argument carries over from one approximation to the next, so that we have in general

$$(2.44) \qquad | x_{\lambda+n}{}^{(k)} - x_{\lambda+n}{}^{(k-1)} | \leqq (1/2^{(k-1)}) \cdot \Lambda\zeta_1{}^{\lambda+n}, \qquad n \geqq M.$$

For fixed n, therefore, $x_{\lambda+n}{}^{(k)}$ has a limit as $k \to \infty$:

$$(2.45) \qquad \lim_{k\to\infty} x_n{}^{(k)} = x_n, \qquad (n = 0, 1, \cdots).$$

(For $n < \lambda + M$, $x_n = X_n$.) This sequence we shall now show is a solution of $\mathfrak{D}^{\dagger(M)}$, or, what is the same thing, of (2.29). For $k < p$,

$$| x_{\lambda+n}{}^{(k)} - x_{\lambda+n}{}^{(p)} | \leqq \Lambda\zeta_1{}^{\lambda+n}\{1/2^k + 1/2^{k+1} + \cdots + 1/2^{p-1}\}.$$

Hence on letting $p \to \infty$:

$$(2.46) \qquad | x_{\lambda+n}{}^{(k)} - x_{\lambda+n} | \leqq (\Lambda/2^{k-1})\zeta_1{}^{\lambda+n}, \qquad n \geqq M.$$

Therefore

$$| \sum_{r=M}^{n} \delta_{n-r} \sum_{s=0}^{\infty} \beta_{rs}\{x_s{}^{(k)} - x_s\} | \leqq (\Lambda/2^{k-1}) \sum_{r=M}^{n} | \delta_{n-r} | \{ \sum_{s=0}^{\infty} | \beta_r | \zeta_1{}^s \}.$$

Since k enters on the right only in the factor 2^{k-1}, we see that for n fixed,

$$\lim_{k\to\infty} \sum_{r=M}^{n} \delta_{n-r} \sum_{s=0}^{\infty} \beta_{rs} x_s{}^{(k)} = \sum_{r=M}^{n} \delta_{n-r} \sum_{s=0}^{\infty} \beta_{rs} x_s;$$

and on letting $k \to \infty$ in (2.40) we obtain (2.29). That is, $\{x_n\}$ is a solution of (2.29).

There remains the matter of type. Since

$$x_{\lambda+n} = x_{\lambda+n}^{(0)} + \sum_{k=1}^{\infty} \{x_{\lambda+n}^{(k)} - x_{\lambda+n}^{(k-1)}\},$$

therefore

$$| x_{\lambda+n} | \leqq \Lambda\zeta_1^{\lambda+n} \sum_{k=1}^{\infty} 2^{1-k} = 2\Lambda\zeta_1^{\lambda+n},$$

so that $((x_n)) \leqq \zeta_1 < \zeta$. We wish to show that we actually have $((x_n)) \leqq 1$. Suppose $((x_n)) = \mu > 1$. Choose a new ζ, say ζ', in $1 < \zeta' < \mu$. Corresponding to this choice we can then determine an index $M' > M$ so that the truncate system $\mathfrak{D}^{\dagger(M')}$ has a unique solution $\{x'_n\}$, of type not exceeding ζ' for which $x'_0, \cdots, x'_{\lambda+M'-1}$ are preassigned. Choose $x'_i = x_i$, $(i = 0, \cdots, \lambda + M' - 1)$, where $\{x_i\}$ satisfies the original truncate $\mathfrak{D}^{\dagger(M)}$. We then have two sets $\{x_n\}$, $\{x'_n\}$ satisfying $\mathfrak{D}^{\dagger(M')}$, and both of type not exceeding ζ. By uniqueness these must be identical. Hence $((x_n)) \leqq \zeta' < \mu$. As this is a contradiction, the assumption that $((x_n)) > 1$ is untenable. That is, the unique solution of $\mathfrak{D}^{\dagger(M')}$ of type not exceeding ζ is in fact of type not exceeding unity. The proof is now complete.

At this point in the Perron system it is possible to determine $x_{M-1}, \cdots, x_0$, uniquely and in that order, in terms of $x_M, \cdots, x_{\lambda+M-1}$, so that the first M equations of the system are also satisfied. This step-by-step ladder is however not available from $\mathfrak{D}^{\dagger(M)}$ to $\mathfrak{D}^{\dagger}$. In fact, the complete system $\mathfrak{D}^{\dagger}$ (or $\mathfrak{D}$) *may fail to have a solution for some sequence* $\{c_n\}$ *of type not exceeding one.*

This is easy to see if the first M linear forms of $\mathfrak{D}$ are linearly dependent, for in this case the corresponding right hand members, that is, the c_n's, cannot be independent; hence sets $c_0, \cdots, c_{M-1}$ will exist for which the first M equations have no solution. A situation like this is entirely consistent with conditions Q. But even if the first M forms are formally linearly independent, it may yet happen that the conditions imposed on $\{x_n\}$ by the truncate system $\mathfrak{D}^{(M)}$ make the first M forms in effect linearly dependent; and if so, constants $c_0, \cdots, c_{M-1}$ can be found for which there will be no solution. We shall presently give an example illustrating this possibility.

LEMMA 4. *Let the conditions and notation of Theorem* 1 *hold. Let* $x_i = X_i$, $(i = 0, \cdots, \lambda + M - 1)$ *be assigned. Then the unique solution* $\{x_n\}$ *of Theorem* 1 *has the form*

$$(2.47) \qquad x_n = H_n + \sum_{i=0}^{\lambda+M-1} H_{ni} X_i, \qquad n \geqq \lambda + M,$$

where all H's are independent of X's, and where H_{ni} *depends only on*

$p_0, \cdots, p_\lambda$ *and* $\{\beta_{ij}\}$, $i \geqq M$, *while* H_n *depends only on these and on* $\{c_i\}$, $i \geqq M$.

To see this, let us return to the successive approximations used in the proof of Theorem 1. We have

$$(2.48) \qquad x_{\lambda+n}{}^{(1)} - x_{\lambda+n}{}^{(0)} = x_{\lambda+n}{}^{(1)} = \nu_{\lambda+n}{}^{(1)} + \sum_{i=0}^{\lambda+M-1} \nu_{\lambda+n,i}{}^{(1)} X_i, \qquad n \geqq M,$$

where

$$(2.49) \qquad \nu_{\lambda+n}{}^{(1)} = c'_n; \qquad \nu_{\lambda+n,i}{}^{(1)} = -\sum_{r=M}^{n} \delta_{n-r}\beta_{ri}, \qquad i < M;$$

$$\nu_{\lambda+n,i}{}^{(1)} = -\sum_{r=0}^{i-M} p_{i-M-r}\delta_{n-M-r} - \sum_{r=M}^{n} \delta_{n-r}\beta_{ri}, \quad M \leqq i \leqq \lambda + M - 1.$$

And by recurrence, for $k > 1$,

$$(2.50) \qquad x_{\lambda+n}{}^{(k)} - x_{\lambda+n}{}^{(k-1)} = \nu_{\lambda+n}{}^{(k)} + \sum_{i=0}^{\lambda+M-1} \nu_{\lambda+n,i}{}^{(k)} X_i, \qquad n \geqq M,$$

where

$$(2.51) \qquad \nu_{\lambda+n}{}^{(k)} = -\sum_{r=M}^{n} \sum_{s=\lambda+M}^{\infty} \delta_{n-r}\beta_{rs}\nu_s{}^{(k-1)},$$

$$\nu_{\lambda+n,i}{}^{(k)} = -\sum_{r=M}^{n} \sum_{s=\lambda+M}^{\infty} \delta_{n-r}\beta_{rs}\nu_{si}{}^{(k-1)}.$$

Hence for $n \geqq M$,

$$(2.52) \qquad x_{\lambda+n} = \sum_{k=1}^{\infty} (x_{\lambda+n}{}^{(k)} - x_{\lambda+n}{}^{(k-1)}) = \sum_{k=1}^{\infty} \{\nu_{\lambda+n}{}^{(k)} + \sum_{i=0}^{\lambda+M-1} \nu_{\lambda+n,i}{}^{(k)} X_i\}.$$

All series involved are absolutely convergent in view of the inequalities obtained in Theorem 1.

COROLLARY. *The coefficients* H_n, H_{ni} *of* (2.47) *are given by*

$$(2.53) \qquad H_n = \sum_{k=1}^{\infty} \nu_n{}^{(k)}, \qquad H_{ni} = \sum_{k=1}^{\infty} \nu_{ni}{}^{(k)}, \qquad n \geqq \lambda + M.$$

Consider now a particular system $\mathfrak{D}^\dagger$, fulfilling the conditions of Theorem 1, in which $\beta_{n0} \neq 0$ for all n. Having determined M, and having assigned the values $x_i = X_i$, $i < \lambda + M$, the quantities H_n, H_{ni} are then determined. If the set $\{x_n\}$ is then substituted into the first equation of $\mathfrak{D}^\dagger$ the following condition is obtained:

$$\sum_{i=0}^{\lambda+M-1} X_i\{\beta_{0i} + (\sum_{s=\lambda+M}^{\infty} \beta_{0s}H_{si}) + p_i\} = \gamma_0 - \sum_{s=\lambda+M}^{\infty} \beta_{0s}H_s$$

where $p_i = 0$ for $i > \lambda$. Now the relations

$$(2.54) \qquad H_{s0} = H_{s1} = \cdots = H_{s,\lambda+M-1} = 0$$

cannot hold for all $s \geqq \lambda + M$. For suppose they do hold. From (2.51) we get, on summing from $k = 2$ to ∞:

$$(H_{\lambda+n,i} - \nu_{\lambda+n,i}{}^{(1)}) + \sum_{r=M}^{n} \sum_{s=\lambda+M}^{\infty} \delta_{n-r}\beta_{rs}H_{si} = 0 \qquad (n \geqq M);$$

or, on setting $y_s = H_{si}$, $s \geqq \lambda + M$, and $\nu_{si}{}^{(1)} = r_s$:

$$(2.55) \qquad y_{\lambda+m} + \sum_{r=M}^{n} \sum_{s=\lambda+M}^{\infty} \delta_{n-r}\beta_{rs}y_s = r_{\lambda+n}, \qquad n \geqq M.$$

If we further define $y_s = 0$, $s < \lambda + M$, then (2.55) is of type (2.29), with $r_{\lambda+n}$ replacing c'_n. As $((r_{\lambda+n})) \leqq 1$ (a fact easily verified), we conclude that (2.55) has a unique solution of type not exceeding one. Now $\{H_{si}\}$, with i fixed, is such a solution. Hence if (2.54) holds, then $r_{\lambda+n} = 0$, $n \geqq M$. That is,

$$\nu_{\lambda+n,i}{}^{(1)} = 0, \qquad (i = 0, 1, \cdots, \lambda + M - 1;\ n = M, M+1, \cdots).$$

Therefore from (2.49),

$$\sum_{r=M}^{n} \delta_{n-r}\beta_{ri} = 0, \qquad i < M.$$

On taking $n = M, M+1, \cdots$ we find, successively, (since $\delta_0 = 1$) that $\beta_{Mi} = 0$, $\beta_{M+1,i} = 0, \cdots$. That is, $\beta_{ni} = 0$ for all $n \geqq M$ and for all i in $0 \leqq i < M$. But our hypothesis is that $\beta_{n0} \neq 0$ for all n. We thus have a contradiction, and conditions (2.54) are untenable.

There exists, therefore, an index $s = s_1 \geqq \lambda + M$, and an i_1 in $0 \leqq i_1 < \lambda + M - 1$, for which $H_{s_1, i_1} \neq 0$. We now revise $\mathfrak{D}^{\dagger}$ by altering its first equation in the following manner: First redefine β_{0,s_1} and β_{0s} ($s \geqq \lambda + M$, $s \neq s_1$) so that

$$\beta_{0,s_1} = 1; \qquad \beta_{0s} = 0, \qquad s \geqq \lambda + M,\ s \neq s_1.$$

The first equation of the new $\mathfrak{D}^{\dagger}$ reduces to

$$(2.56) \qquad \sum_{i=0}^{\lambda+M-1} (\beta_{0i} + \rho_i + H_{s_1,i})X_i = \gamma_0 - H_{s_1}.$$

Now redefine β_{0i}, $i < \lambda + M$, so that

$$\beta_{0i} + p_i + H_{s_1,i} = 0, \qquad i < \lambda + M.$$

Since $H_{s_1, i_1} \neq 0$, it follows that for $i = i_1$ the quantity $\beta_{0i} + p_i$ is not zero. This assures us that the first equation in the revised system $\mathfrak{D}^\dagger$ does not have all of its coefficients zero. Now, however, the left side of (2.56) is zero for all choices of X_i, whereas the right hand side, involving as it does the quantity γ_0, need not be zero. Consequently the system $\mathfrak{D}^\dagger$ does not always have a solution. Moreover, the $M - 1$ remaining equations of the first M can be redefined so that there is *formal* linear independence.

To summarize, we have

THEOREM 2. *There exist systems $\mathfrak{D}$ with the following properties:*

(i) *Conditions Q hold.*

(ii) *Every finite set of linear forms of $\mathfrak{D}$ is formally linearly independent.*

(iii) *For at least one sequence $\{c_n\}$ with $((c_n)) \leqq 1$, (in fact, "in general"), there is no solution $\{x_n\}$ of $\mathfrak{D}$ for which $((x_n)) \leqq 1$.*

In view of Theorem 2, it is desirable to know when a system $\mathfrak{D}$ will have a solution $\{x_n\}$ for every $\{c_n\}$. Let the conditions of Theorem 1 hold, so that $\mathfrak{D}^{\dagger(M)}$ has a unique solution for preassigned $x_i = X_i$, $i < \lambda + M$. If we substitute the values x_n as given by (2.47) into the first M equations of $\mathfrak{D}^\dagger$ there result the relations

$$(2.57) \qquad \sum_{s=0}^{\lambda+M-1} X_s\{\beta_{is} + p_{s-i} + \sum_{r=\lambda+M}^{\infty} \beta_{ir} H_{rs}\} = \gamma_i - \sum_{r=\lambda+M}^{\infty} \beta_{ir} H_r, \qquad i < M,$$

where $p_k = 0$ unless $k = 0, 1, \cdots, \lambda$.

Define A_{is} by

$$(2.58) \qquad A_{is} = \beta_{is} + p_{s-i} + \sum_{r=\lambda+M}^{\infty} \beta_{ir} H_{rs}, \qquad \left\{\begin{array}{l} i = 0, 1, \cdots, M-1; \\ s = 0, 1, \cdots, \lambda + M - 1. \end{array}\right.$$

Then we can state

THEOREM 3. *Let the system $\mathfrak{D}$ (or $\mathfrak{D}^\dagger$) satisfy conditions Q, and let M be chosen as in Theorem 1. A necessary and sufficient condition that $\mathfrak{D}$ have a solution $\{x_n\}$ with $((x_n)) \leqq 1$ for every sequence $\{c_n\}$ for which $((c_n)) \leqq 1$ is that the matrix*

$$\left\| \begin{array}{ccc} A_{00} & \cdots & A_{0,\lambda+M-1} \\ \cdot & \cdots & \cdot \\ A_{M-1,0} & \cdots & A_{M-1,\lambda+M-1} \end{array} \right\|$$

be of rank M; and in this case there exist indices j_r, $(r = 1, \cdots, \lambda)$ in the

range $0 \leqq j_1 < j_2 < \cdots < j_\lambda \leqq \lambda + M - 1$ *and independent of* $\{c_n\}$, *such that on preassigning the values* $x_{j_r} = X_{j_r}$, $(r = 1, \cdots, \lambda)$, *there is one and only one solution* $\{x_n\}$ *of* $\mathfrak{D}$ *with* $((x_n)) \leqq 1$.

The proof is obvious.

We turn now to the *homogeneous system.* If $c_n \equiv 0$, then $\gamma_n \equiv 0$, so that from (2.27), $c'_n \equiv 0$. This implies that $\nu_{\lambda+n}^{(k)} \equiv 0$, $(k = 1, 2, \cdots)$, and therefore $H_n \equiv 0$. Let $x_i = X_i$, $(i = 0, \cdots, \lambda + M - 1)$ be given, and let x_n satisfy the M-th truncate (homogeneous) system. The relation (2.47) reduces to

$$x_n = \sum_{i=0}^{\lambda+M-1} H_{ni} X_i, \qquad n \geqq \lambda + M; \tag{2.59}$$

and (2.57) to

$$\sum_{s=0}^{\lambda+M-1} X_s \{\beta_{is} + p_{s-i} + \sum_{r=\lambda+M}^{\infty} \beta_{ir} H_{rs}\} = 0, \tag{2.60}$$

that is, to

$$\sum_{s=0}^{\lambda+M-1} A_{is} X_s = 0, \qquad (i = 0, 1, \cdots, M - 1). \tag{2.61}$$

The homogeneous system has a solution if and only if (2.61) can be satisfied by suitable choice of $X_0, \cdots, X_{\lambda+M-1}$. We accordingly have

THEOREM 4. *Let the homogeneous system* $\mathfrak{D}$ (*or* $\mathfrak{D}^\dagger$) $[c_n \equiv \gamma_n \equiv 0]$ *satisfy conditions* Q, *and let* M *be chosen as in Theorem* 1. *There exist precisely* $\lambda + M - q$ *linearly independent solutions of type not exceeding one, where* q *is the rank of the A-matrix of Theorem* 3. *In particular, since* $q \leqq M$, *there exist at least* λ *linearly independent solutions, and if* $\lambda \geqq 1$, *there is at least one solution that is not identically zero.*

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POWER SERIES WITH BOUNDED COEFFICIENTS.*

By R. J. DUFFIN and A. C. SCHAEFFER.

The first part of the present paper concerns entire functions of exponential type; the second part deals with applications of these results to power series which are bounded in a sector of the unit circle.[1]

PART I.

1. We shall say that an entire function $f(z)$ is of exponential type k if there is a constant A such that

$$|f(z)| \leqq Ae^{k|z|}.$$

It was shown by Cartwright[2] that if an entire function of exponential type $k < \pi$ is bounded at the integer points $0, \pm 1, \cdots$ then it is bounded on the entire real axis. We shall be concerned with entire functions $f(z)$ which are bounded at a sequence of points $\{\lambda_n\}$. Boas[3] showed that if $|\lambda_n - n| < L < 1/(2\pi^2)$, $(n = 0, \pm 1, \cdots)$ then $f(z)$ is bounded for real z. The condition we impose on the sequence $\{\lambda_n\}$ is that there be two constants Γ and γ such that

$$(1) \qquad |\lambda_n - n| \leqq \Gamma, \qquad (n = 0, \pm 1, \pm 2, \cdots),$$

$$(2) \qquad |\lambda_n - \lambda_m| \geqq \gamma > 0, \qquad n \neq m.$$

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* Part II received May 2, 1942; Revised and Augmented by Part I, August 31, 1944.

[1] The authors are indebted to the referee, who corrected an error in the original manuscript, and to the editor, who suggested several valuable improvements in the proof.

[2] M. L. Cartwright, "On certain integral functions of order one," *Quarterly Journal of Mathematics* (Oxford Series), vol. 7 (1936), pp. 46-55; A. Pfluger, "On analytic functions bounded at the lattice points," *Proceedings of the London Mathematical Society*, (2), vol. 42 (1936-37), pp. 305-315; A. J. Macintyre, "Laplace's transformation and integral functions," *Proceedings of the London Mathematical Society*, (2), vol. 45 (1938-39), pp. 1-20; R. P. Boas, Jr., "Entire functions bounded on a line," *Duke Mathematical Journal*, vol. 6 (1940), pp. 148-169; R. P. Boas, Jr. and A. C. Schaeffer, "A theorem of Cartwright," *Duke Mathematical Journal*, vol. 9 (1942), pp. 879-883.

[3] R. P. Boas, Jr., *loc. cit.*

2. THEOREM I. *Let* $\{\lambda_n\}$ *be a sequence which satisfies* (1) *and* (2). *If* $f(z)$ *is an entire function which satisfies*

$$(3) \qquad |f(z)| \leqq Ae^{k|z|}, \qquad k < \pi,$$

$$(4) \qquad |f(\lambda_n)| \leqq 1, \qquad (n = 0, \pm 1, \cdots)$$

then

$$(5) \qquad |f(z)| \leqq N(\Gamma, \gamma, k)e^{k|y|}.$$

This theorem is a consequence of a somewhat similar result concerning functions which are regular and of exponential growth in the half-plane $\mathcal{R}(z) \geqq 0$. In this case we suppose that the function is bounded at a sequence of points $\{\lambda_n\}$ which satisfies:

$$(6) \qquad |\lambda_n - n| \leqq \Gamma, \qquad n > \Gamma,$$

$$(7) \qquad |\lambda_n - \lambda_m| \geqq \gamma > 0, \qquad n \neq m.$$

Here Γ and γ are two positive constants, and we suppose that λ_n is defined only for integers $n > \Gamma$. It then follows that all the points of the sequence $\{\lambda_n\}$ lie in the half-plane $\mathcal{R}(z) > 0$.

THEOREM II. *Let* $\{\lambda_n\}$ *be a sequence which satisfies* (6) *and* (7). *If* $f(z)$ *is regular in the half-plane* $\mathcal{R}(z) \geqq 0$ *and*

$$(8) \qquad |f(z)| \leqq e^{k|z|}, \qquad k < \pi,$$

$$(9) \qquad |f(\lambda_n)| \leqq 1, \qquad n > \Gamma,$$

then

$$(10) \qquad |f(z)| \leqq M(\Gamma, \gamma, k)e^{k|y|}, \qquad x \geqq 0.$$

3. Although in Theorem I the constant A of inequality (3) does not influence the dominant $Ne^{k|y|}$ which is obtained for $f(z)$, the analogous statement is not true in Theorem II. We could not replace inequality (8) by an inequality of the form

$$|f(z)| \leqq Ae^{k|z|}, \qquad k < \pi,$$

without changing the inequality (10). This is shown by the example

$$f(z) = \prod_{\nu=0}^{m} (z - \nu) \prod_{\nu=1}^{m+1} (z + \nu)^{-1} e^{m-z}.$$

This function is regular and bounded in the right half-plane, and it is bounded by 1 at the integer points, $z = 0, 1, 2, \cdots$. However, $|f(\tfrac{1}{2})|$ $e^{m-\frac{1}{2}}(2m+3)^{-1}(2m+1)^{-1}$, which can be made arbitrarily large by choosing

m large. The following corollary of Theorem II is in some respects a closer analogue of Theorem I than is Theorem II.

COROLLARY I. *Let* $\{\lambda_n\}$ *be a sequence which satisfies* (6) *and* (7). *If* $f(z)$ *is regular in* $\mathfrak{R}(z) \geqq 0$ *and*

(11) $$|f(z)| \leqq Ae^{k|z|}, \qquad k < \pi,$$

(12) $$|f(\lambda_n)| \leqq 1, \qquad n > \Gamma,$$

then

(13) $$|f(x)| \leqq N(\Gamma, \gamma, k), \qquad x \geqq A^2/(\pi - k).$$

We shall also be interested in functions which tend toward zero as z becomes infinite through a suitable sequence of values. The following corollary is a generalization of a result of Young [4] who proved that if

$$f(z) = \int_a^b e^{izt} dg(t), \qquad -\pi < a < b < \pi,$$

where $g(t)$ is of bounded variation and $f(n) \to l$ as $n \to \pm\infty$ then $f(x) \to l$ as $x \to \pm\infty$.

COROLLARY II. *Let* $\{\lambda_n\}$ *be a sequence which satisfies* (6) *and* (7). *If* $f(z)$ *is regular in* $\mathfrak{R}(z) \geqq 0$ *and*

(14) $$|f(z)| \leqq Ae^{k|z|}, \qquad k < \pi,$$

(15) $$f(\lambda_n) = o(1), \qquad n \to \infty,$$

then

(16) $$f(x) = o(1), \qquad x \to \infty.$$

4. Since we shall have several occasions to use the *Phragmén-Lindelöf* principle in various forms we state a lemma which will cover the several cases.

LEMMA I. *If* $f(z)$ *is regular in* $\mathfrak{R}(z) \geqq 0$ *and*

(17) $$|f(z)| \leqq Ae^{k|z|}, \qquad \mathfrak{R}(z) \geqq 0,$$

(18) $$|f(x)| \leqq B, \qquad x \geqq 0,$$

where $B \geqq A$ *then*

(19) $$|f(z)| \leqq Be^{k|y|}, \qquad \mathfrak{R}(z) \geqq 0.$$

If $f(z)$ *is an entire function and*

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[4] R. C. Young, "The asymptotic behavior of $F(z) = \int_{a-0}^{b+0} e^{zt} dg(t)$," *Mathematische Zeitschrift*, vol. 40 (1936), pp. 292-311.

$$f(z) = O(e^{k|z|}) \tag{20}$$

$$|f(x)| \leqq B \tag{21}$$

then

$$|f(z)| \leqq Be^{k|y|}. \tag{22}$$

The second part of this lemma has been proved by Pólya and Szegö[5] using a method which is different from the one employed here. If $f(z)$ satisfies (20) and (21) the function $f(z)e^{ikz}$ is bounded on the positive halves of the real and imaginary axis and is of order one or less in the angular region between them. The Phragmén-Lindelöf principle then shows that it is bounded in the first quadrant. Likewise it is bounded in the second quadrant. Hence

$$|f(z)e^{ikz}| \leqq C$$

in the upper half-plane $y \geqq 0$, where C is some constant. But $f(z)e^{ikz}$ is bounded by B on the real axis so applying the Phragmén-Lindelöf principle to this function with the angular region now being the upper half-plane we find that

$$|f(z)e^{ikz}| \leqq B, \qquad y \geqq 0.$$

Likewise we find that the function $f(z)e^{-ikz}$ is bounded by B in the lower half-plane. This proves (22).

The inequality (19) is proved by applying the Phragmén-Lindelöf principle to the functions $f(z)e^{ikz}$ and $f(z)e^{-ikz}$ in the first and fourth quadrants respectively.

5. In the proof of Theorem II we shall make use of Lemma II, which is weaker than Theorem II, but contains the essential step in the proof of that Theorem.

LEMMA II. *Let* $\{\lambda_n\}$ *be a sequence which satisfies* (6) *and* (7). *If* $f(z)$ *is regular in the half-plane* $\mathfrak{R}(z) \geqq 0$ *and*

$$|f(z)| \leqq e^{k|z|}, \qquad k < \pi, \tag{23}$$

$$|f(\lambda_n)| \leqq 1, \qquad n > \Gamma, \tag{24}$$

$$f(x) = O(1), \qquad x \to \infty, \tag{25}$$

then

$$|f(x)| \leqq M(\Gamma, \gamma, k), \qquad x \geqq 0. \tag{26}$$

[5] Pólya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, p. 36, problem 202.

The point of this lemma is that if Γ, γ, and k are fixed then the functions which satisfy the conditions of Theorem II may be divided into two classes, namely, (1) those which are unbounded on the positive real axis, and (2) those which are bounded on the positive real axis. The lemma states that all functions of the second class are bounded by a common constant M. We shall show that every function which satisfies the conditions of Theorem II is the limit of a sequence of functions of the second class, and is therefore bounded on the positive real axis. The class (1) is therefore non-existant.

Proof of Lemma II. Let the constants Γ, γ, and k be fixed, and let $f_\nu(z)$, $(\nu = 1, 2, 3, \cdots)$ be any sequence of functions which satisfy the conditions of Lemma II. Let

$$c_\nu = \sup_{0 \leqq x < \infty} |f_\nu(x)|. \tag{27}$$

Now each c_ν is finite according to hypothesis, and it is to be shown that the c_ν are bounded in their set. We suppose the sequence $f_\nu(z)$ chosen such that $c_\nu > \nu$, and show that this leads to a contradiction. Each $f_\nu(z)$ is bounded by 1 at a sequence $\{\lambda_n^{(\nu)}\}$ which satisfies (6) and (7). If we combine (23) and (27) with Lemma I we find that

$$|f_\nu(z)| \leqq c_\nu e^{k|y|}.$$

There will be some point at which $|f_\nu(x)|$ is near its upper bound, let x_ν be chosen such that

$$|f_\nu(x_\nu)| \geqq c_\nu(1 - 1/\nu).$$

In view of (23) it follows that $x_\nu \to \infty$ as ν does.

Let

$$\phi_\nu(z) = f_\nu(z + [x_\nu])/c_\nu$$

where $[x_\nu]$ is the integer part of x_ν. Then $\phi_\nu(z)$ is regular in the half plane $\Re(z) \geqq -[x_\nu]$ and it satisfies

$$|\phi_\nu(z)| \leqq e^{k|y|}, \qquad x \geqq -[x_\nu],$$

$$\max_{0 \leqq x \leqq 1} |\phi_\nu(x)| \geqq 1 - 1/\nu$$

$$|\phi_\nu(\mu_n^{(\nu)})| \leqq 1/c_\nu, \qquad n > \Gamma - [x_\nu].$$

Here $\{\mu_n^{(\nu)}\}$ is a sequence which is obtained by translating the sequence $\{\lambda_n^{(\nu)}\}$ a distance $[x_\nu]$ to the left.

6. Since the functions $\phi_\nu(z)$ are dominated by $e^{k|y|}$ they are equicontinuous in any given bounded region for large ν, and so Ascoli's convergence

theorem concerning bounded equicontinuous functions shows that there is a subsequence which tends to a limit $\phi(z)$ which is an entire function and satisfies

$$(28) \qquad |\phi(z)| \leqq e^{k|y|}; \qquad \max_{0 \leqq x \leqq 1} |\phi(x)| = 1.$$

Moreover, the sequence may be so chosen that, for each n, $\mu_n^{(\nu)}$ tends to a limit as ν becomes infinite. Then

$$(29) \qquad \phi(\mu_n) = 0, \qquad (n = 0, \pm 1, \cdots)$$

and the sequence $\{\mu_n\}$ satisfies

$$| \mu_n - n | \leqq \Gamma, \qquad (n = 0, \pm 1, \pm 2, \cdots)$$
$$| \mu_n - \mu_m | \geqq \gamma > 0, \qquad n \neq m.$$

But Jensen's formula shows that if $\phi(z)$ is a function which satisfies (28) then

$$\overline{\lim_{r\to\infty}}\, r^{-1} \int_1^r n(t) t^{-1} dt \leqq 2k/\pi < 2$$

where $n(t)$ is the number of zeros of $\phi(z)$ in $|z| \leqq t$. But from (29) it is easily seen that

$$\underline{\lim_{r\to\infty}}\, r^{-1} \int_1^r n(t) t^{-1} dt \geqq 2.$$

This contradiction proves that there exists a constant $M(\Gamma, \gamma, k)$ such that the inequality (26) is valid.

7. Proof of Theorem II. Let $f(z)$ satisfy the conditions of Theorem II and let

$$\rho = \overline{\lim_{x\to\infty}}\, x^{-1} \log |f(x)|.$$

We choose a constant a subject to the condition that $a > (\rho + |\rho|)/2$. Then let $\epsilon > 0$ be chosen such that $\epsilon < a$ and $\epsilon < (\pi - k)/2$. For $(\nu = 1, 2, 3, \cdots)$ let

$$g_\nu(z) = f(z) e^{-az} \sum_{p=0}^{\nu} (\epsilon z)^p / p!.$$

The sum is no greater in modulus than $e^{\epsilon|z|}$ since it is dominated term by term by the Taylor's expansion of $e^{\epsilon|z|}$ about the origin. We then see that in the half-plane $\mathcal{R}(z) \geqq 0$,

$$| g_\nu(z) | \leqq | f(z) | \, e^{\epsilon|z|} \leqq e^{(k+\epsilon)|z|} \leqq e^{\kappa|z|}$$

where we define κ as $(k + \pi)/2 < \pi$. Also if we write $\lambda_n = \mu_n + i\tau_n$ then

$$| g_\nu(\lambda_n) | \leqq e^{-a\mu_n} e^{\epsilon|\lambda_n|} \leqq e^{\epsilon\Gamma}$$

since $\epsilon | \lambda_n | - a\mu_n \leqq \epsilon(\mu_n + | \tau_n |) - \epsilon\mu_n = \epsilon | \tau_n | \leqq \epsilon\Gamma$.

Moreover, $g_\nu(x)$ tends to zero as x becomes infinite with a, ϵ, and ν fixed, and, in particular, $g_\nu(x)$ is bounded on the positive real axis. We see from the above remarks that the function $e^{-\epsilon\Gamma} g_\nu(z)$ satisfies the conditions of Lemma II with k replaced by κ. It therefore follows that

$$(30) \qquad | g_\nu(x) | \leqq e^{\epsilon\Gamma} M(\Gamma, \gamma, \kappa), \qquad x \geqq 0.$$

Now keeping a and ϵ fixed and letting ν become infinite we see that the inequality (30) must also be satisfied by the limit $f(z)e^{-az}e^{\epsilon z}$ of $g_\nu(z)$. Thus

$$(31) \qquad | f(x) | \leqq e^{(a-\epsilon)x} e^{\epsilon\Gamma} M(\Gamma, \gamma, \kappa), \qquad x \geqq 0.$$

We now distinguish two cases, $\rho > 0$ and $\rho \leqq 0$. If $\rho > 0$ we choose an ϵ, $0 < \epsilon < \rho$, of course keeping the previous condition $\epsilon < (\pi - k)/2$. Now keeping ϵ fixed and letting a approach ρ we see that

$$| f(x) | \leqq e^{(\rho-\epsilon)x} e^{\epsilon\Gamma} M(\Gamma, \gamma, \kappa)$$

on the positive real axis. But this contradicts the definition of ρ and shows that $\rho \leqq 0$.

Since $\rho \leqq 0$ we let a and ϵ each approach zero. Then the inequality (31) shows that $f(x)$ is bounded by $M(\Gamma, \gamma, \kappa)$. Since $f(x)$ is bounded by some constant we may now apply Lemma II, and it follows that $f(x)$ is bounded by $M(\Gamma, \gamma, k)$ for $x \geqq 0$. Now if we apply Lemma I to $f(z)$ we see, since clearly $M \geqq 1$, that the inequality (10) is valid throughout the right half-plane.

8. To prove Corollary I it is sufficient to consider the case in which the constant A of inequality (11) is greater than 1, since if $A \leqq 1$ the result is clear from Theorem II. Let $\beta = 2(\log A)/(\pi - k)$ and $\tau = \beta A$. The function

$$g(z) = z(z + \tau)^{-1} f(z)$$

satisfies in the right half-plane $\mathcal{R}(z) \geqq 0$ the inequalities

$$| g(z) | \leqq \beta\tau^{-1} | f(z) | \leqq A^{-1} A e^{k|z|} = e^{k|z|}, \qquad | z | \leqq \beta,$$

and

$$| g(z) | \leqq | f(z) | \leqq A e^{k|z|} \leqq e^{\kappa|z|}, \qquad | z | \geqq \beta,$$

where $\kappa = (\pi + k)/2 < \pi$. Here we have used the fact that $| z(z + \tau)^{-1} | \leqq \beta\tau^{-1} = A^{-1}$ in the semicircle $| z | \leqq \beta$, $\mathcal{R}(z) \geqq 0$, and $| z(z + \tau)^{-1} | \leqq 1$ in the half-plane $\mathcal{R}(z) \geqq 0$.

It is thus seen that $g(z)$ satisfies the conditions of Theorem II with k replaced by κ, and so

$$|g(x)| \leqq M(\Gamma, \gamma, \kappa), \qquad x \geqq 0.$$

Then expressing $f(z)$ is terms of $g(z)$ and using the last inequality we see that

$$|f(x)| \leqq (x+\tau)x^{-1}M(\Gamma, \gamma, \kappa) \leqq 2M(\Gamma, \gamma, \kappa)$$

when $x \geqq \tau = \beta A = 2A(\log A)/(\pi - k)$. Since $2 \log A < A$ the result follows if we write $N(\Gamma, \gamma, k)$ in place of $2M(\Gamma, \gamma, \pi/2 + k/2)$.

9. Proof of Corollary II. If $f(z)$ satisfies the conditions of Corollary II let

$$|f(\lambda_n)| < \epsilon, \qquad n \geqq m,$$

where ϵ is some given positive number. If p is any positive integer the function

$$z^p(z+1)^{-p}f(z)$$

will be bounded by ϵ at the points $z = \lambda_n$ for $n \geqq m$. If p is made sufficiently large this function will also be bounded by ϵ at the points $z = \lambda_n$ for $n < m$. Choosing p sufficiently large, the function

$$g(z) = \epsilon^{-1}z^p(z+1)^{-p}f(z)$$

is bounded by 1 at the points $z = \lambda_n$ for all n in the range $n > \Gamma$. Since, clearly,

$$|g(z)| \leqq A\epsilon^{-1}e^{k|z|},$$

Corollary I states that $g(x)$ is bounded by $N(\Gamma, \gamma, k)$ for all large x. Then, expressing $f(z)$ is terms of $g(z)$,

$$f(z) = \epsilon(z+1)^p z^{-p} g(z),$$

we see that for all sufficiently large x,

$$|f(x)| < 2\epsilon N(\Gamma, \gamma, k).$$

10. Proof of Theorem I. If $f(z)$ satisfies the conditions of Theorem I then $f(z)$ and $f(-z)$ each satisfy the conditions of Corollary I, from which we infer that $f(z)$ is bounded by some constant on the real axis. If we use the second part of Lemma I we see that

$$|f(z)| \leqq Ce^{k|y|} \tag{32}$$

where C is some constant. It is then only necessary to show that the constant C can be chosen to depend exclusively on Γ, γ, and k.

From inequality (32) we see that if m is any positive integer the function

$$g_m(z) = f(z - m)$$

satisfies the inequality (11) with $A = C$. It is also bounded at a sequence of points $\{\mu_n\}$ which is obtained by translating the sequence $\{\lambda_n\}$ a distance m to the right. Corollary I therefore shows that

$$|g_m(x)| \leqq N(\Gamma, \gamma, k), \qquad x \geqq C^2/(\pi - k).$$

This in turn shows that $f(x) = g_m(x + m)$ is bounded by $N(\Gamma, \gamma, k)$ in the range $x \geqq -m + C^2/(\pi - k)$. Since m is an arbitrary positive integer it follows that $f(z)$ is bounded by $N(\Gamma, \gamma, k)$ on the entire real axis. If we again apply the second part of Lemma I to the function $f(z)$ we obtain the inequality (5).

11. In the second part of the present paper we shall be concerned with functions which are known to be bounded at all integer points save possibly those of a certain subsequence. Let $\{\mu_n\}$, $(n = 0, \pm 1, \cdots)$, be a sequence of positive and negative integers such that

$$\mu_{n+1} - \mu_n > 0 \tag{33}$$

and

$$\mu_{n+1} - \mu_n \to \infty, \qquad n \to \pm \infty. \tag{34}$$

THEOREM III. *Let $\{\mu_n\}$ be a sequence of integers which satisfies* (33) *and* (34). *If $f(z)$ is an entire function which satisfies*

$$|f(z)| \leqq Ae^{k|z|}, \qquad k < \pi,$$

$$|f(m)| \leqq 1, \qquad m \notin \{\mu_n\},$$

then $f(z)$ is uniformly bounded on the real axis, and

$$|f(z)| \leqq Ee^{k|y|} \tag{35}$$

If, in addition, $f(m) = o(1)$ as $m \to \pm \infty$, $m \notin \{\mu_n\}$, then $f(x) = o(1)$ as $x \to \pm \infty$.

Proof. The function $f(\rho z)$ where ρ is chosen in the range

$$1 < \rho < \pi/k$$

is bounded at the points

$$z_m = m/\rho, \qquad m \notin \{\mu_n\}.$$

Since ρ is chosen greater than 1 it is clear that there exists a subsequence $\{\lambda_n\}$ of the points z_m such that

$$|\lambda_n - n| \leqq \Gamma, \qquad (n = 0, \pm 1, \cdots),$$

where Γ is some constant. Also $|\lambda_n - \lambda_m| \geqq 1/\rho > 0$ when $n \neq m$. Since ρ is chosen less than π/k the function $f(\rho z)$ is of exponential type $k\rho < \pi$. Theorem I now shows that the function $f(\rho z)$, and hence $f(z)$, is bounded on the real axis. Inequality (35) then follows from Lemma I.

If $f(m) = o(1)$ as $m \to \pm\infty$, $m \notin \{\mu_n\}$, then by applying Corollary II to the functions $f(-\rho z)$ and $f(\rho z)$ we see that $f(x) = o(1)$ as $x \to \pm\infty$.

PART II.

1. A well known theorem of Szegö [6] states the following.

Let $f(z) = \sum_0^\infty a_n z^n$ where the coefficients a_n take only a finite number of different values. If $f(z)$ is continuable beyond the unit circle then it is a rational function.

We shall prove the following generalization of this theorem:

THEOREM I. *Let $f(z) = \sum_0^\infty a_n z^n$ where the coefficients a_n take only a finite number of different values. If $f(z)$ is bounded in a sector of the unit circle then it is a rational function.*

2. Suppose now that

$$f(z) = \sum_0^\infty a_n z^n$$

where the series converges in the unit circle $|z| < 1$ and the coefficients a_n are integers. It is known that if such a function is continuable beyond the unit circle then it is a rational function. This is a theorem which was conjectured by Pólya and was first proved by Carlson.[7] In view of the obvious similarity of these two theorems it is natural to ask if there is a corresponding generalization of the Pólya-Carlson theorem. That is, if $f(z)$ is bounded in a sector of the unit circle is it necessarily a rational function? The answer to this question is no, as is shown by the example

$$f(z) = \sum_1^\infty (1-z)^n z^{n!}.$$

This function is regular in the unit circle and its Taylor's expansion

[6] G. Szegö, "Tschebyscheffsche Polynome und nichfortsetzbare Potenzreihen," *Mathematische Annalen*, vol. 87 (1922), pp. 90-111.

[7] F. Carlson, "Über Potenzreihen mit ganzzahligen Koeffizienten, *Mathematische Zeitschrift*, vol. 9 (1921), pp. 1-13.

$$f(z) = \sum_0^\infty a_n z^{\lambda_n}$$

has integer coefficients. Moreover $f(z)$ is bounded in the sector of the unit circle $|\operatorname{amp} z| \leqq \alpha < \pi/3$. But $\lambda_n/n \to \infty$ so the Fabry gap theorem shows that it has the unit circle for a natural boundary and, in particular, is not a rational function.

3. Let $f(z)$ satisfy the conditions of Theorem I and suppose that $f(re^{i\theta})$ is bounded in the sector $\alpha < \theta < \beta$, $0 \leqq r < 1$. It is sufficient to consider the case in which $f(0) = 0$. Since $f(ze^{ip/q})$ will have only a finite number of different coefficients if p and q are integers, we also suppose that $\alpha < \pi < \beta$. We choose an $\epsilon > 0$ such that $\pi - \epsilon > \alpha$ and $\pi + \epsilon < \beta$, and then a $\delta > 0$ such that

$$\epsilon + \log(1 - \delta) > 0. \tag{1}$$

Let Γ be an open contour consisting of the circular arc $|\omega| = 1 - \delta$, $|\operatorname{amp} \omega| \leqq \pi - \epsilon$ and the two radial segments $1 - \delta \leqq |\omega| \leqq 1$, $\operatorname{amp} \omega = \pm (\pi - \epsilon)$. Then for any r in the range $1 - \delta < r < 1$,

$$a_n = \frac{1}{2\pi i} \int_{\Gamma_r} f(\omega)\omega^{-n-1} d\omega + \frac{1}{2\pi} \int_{\pi-\epsilon}^{\pi+\epsilon} f(re^{i\theta}) r^{-n} e^{-in\theta} d\theta$$

where Γ_r denotes the portion of Γ that lies in the circle $|\omega| \leqq r$. This relation holds for negative n if we define a_n as 0 for $n = -1, -2, -3, \cdots$.

The last integral may be written in the form

$$\frac{1}{2\pi i} \int_{\pi-\epsilon}^{\pi+\epsilon} r^{-n} e^{-in\theta} \frac{dF(re^{i\theta})}{d\theta} d\theta = \frac{1}{2\pi i} \int_{\pi-\epsilon}^{\pi+\epsilon} r^{-n} e^{-in\theta} dF(re^{i\theta})$$

where

$$F(\omega) = \int_0^\omega f(t) t^{-1} dt.$$

Since $f(\omega)$ is bounded inside Γ it is clear that $F(\omega)$ satisfies a uniform Lipschitz condition inside Γ, $|F(\omega_1) - F(\omega_2)| \leqq M |\omega_1 - \omega_2|$ when ω_1 and ω_2 lie inside Γ. Thus, as $r \to 1$, $F(re^{i\theta})$ tends uniformly to a limit $F(e^{i\theta})$ in the range $\pi - \epsilon \leqq \theta \leqq \pi + \epsilon$. Letting r appproach 1 we obtain

$$a_n = \frac{1}{2\pi i} \int_\Gamma f(\omega)\omega^{-n-1} d\omega + \frac{1}{2\pi i} \int_{\pi-\epsilon}^{\pi+\epsilon} e^{-in\theta} dF(e^{i\theta}). \tag{2}$$

Since $F(e^{i\theta})$ satisfies a uniform Lipschitz condition the last integral is $o(1)$ as $n \to \pm \infty$.

Let

$$\phi(z) = \frac{1}{2\pi i} \int_\Gamma f(\omega)\omega^{-1} e^{-z \log \omega} d\omega \tag{3}$$

where $\log \omega$ has its principal value. For integer values of z we infer from (2) that

$$(4) \qquad \phi(n) = a_n + o(1), \qquad n \to \pm \infty.$$

Furthermore, $f(\omega)$ is bounded on Γ, and, from (1),

$$|\log \omega| \leqq \pi - \epsilon - \log(1-\delta) < \pi, \qquad \omega \in \Gamma;$$

hence we infer that $\phi(z)$ is an entire function of exponential type less than π,

$$|\phi(z)| \leqq A_1 e^{k|z|}, \qquad k < \pi.$$

Now (4) implies that $\phi(z)$ is bounded at the integer points, so from Theorem I of Part I we obtain the inequality

$$(5) \qquad |\phi(z)| \leqq A_2 e^{k|y|}.$$

This inequality shows that if $\{\mu_n\}$ is any sequence of real numbers the functions $\phi(z-\mu_n)$ will be uniformly bounded in every finite region, and so, by Ascoli's convergence theorem concerning equicontinuous functions, there will be a subsequence which converges to an entire function which satisfies (5). The family of functions $\phi(z-\mu_n)$, μ_n real, is a *normal family.*

4. Let $b_1, b_2, \cdots, b_m$ be the different values that a_n can have,

$$|b_i - b_j| \geqq b > 0, \qquad i \neq j.$$

If r is a given positive integer there will be positive integers $p = p(r)$ and $q = q(r)$, $p > q \geqq 0$, such that

$$a_{p+\nu} = a_{q+\nu}, \qquad 0 \leqq \nu \leqq r.$$

Then either

$$\text{(i)} \qquad a_{p+\nu} = a_{q+\nu}, \qquad \nu \geqq 0$$

or

(ii) there is an integer $R > r$ such that

$$(6) \qquad \begin{aligned} a_{p+\nu} &= a_{q+\nu}, \qquad 0 \leqq \nu \leqq R-1 \\ a_{p+R} &\neq a_{q+R}, \end{aligned}$$

Suppose that there are arbitrarily large integers r for which case (ii) is true. It is to be shown that this leads to a contradiction. Let

$$g_r(z) = \phi(z+p+R) - \phi(z+q+R)$$

where p, q, and R are functions of r which satisfy (6) and $p > q \geqq 0$. Then $g_r(z)$ is an entire function and, in virtue of (5),

$$|g_r(z)| \leqq 2A_2 e^{k|y|}.$$

As r becomes large R does also, so relation (4) shows that

$$(7) \qquad g_r(m) = a_{m+p+R} - a_{m+q+R} + o(1), \qquad r \to \infty,$$

for each fixed integer m. But as r becomes large some subsequence of the functions $g_r(z)$ tends to a limit $g(z)$ which is an entire function and satisfies

(a) $\quad |g(z)| \leqq 2A_2 e^{k|y|}, \qquad k < \pi,$

(b) $\quad |g(0)| \geqq b > 0,$

(c) $\quad g(m) = 0, \qquad (m = -1, -2, \cdots).$

Properties (b) and (c) are obtained from (6) and (7). But a theorem of Carlson states that a function which satisfies (a) and (c) must vanish identically. This contradicts (b).

It has thus been shown that there is an integer for which case (i) is true, and this clearly implies that $f(z)$ is a rational function of the form

$$f(z) = P(z)/(1 - z^{p-q})$$

where $P(z)$ is a polynomial.

5. The methods of the last paragraphs can be used to obtain a theorem concerning power series in which all the coefficients are bounded save possibly those of a Faber sequence. The following theorem is allied to a result of Paley and Wiener,[8] although it does not contain their theorem, nor is it contained in their theorem (for a more general result cf. **6**).

THEOREM II. *Let* $f(z) = \sum_0^\infty a_n z^n$ *be bounded in a sector of the unit circle. Let* $\{n_\kappa\}$ *be a sequence of integers such that*

$$n_{\kappa+1} - n_\kappa \to \infty, \qquad \kappa \to \infty.$$

If the coefficients a_n *are bounded, save possibly when* $n \in \{n_\kappa\}$, *then in fact all coefficients are bounded. If* $a_n = o(1)$ *as* $n \to \infty$, $n \notin \{n_\kappa\}$, *then all coefficients are* $o(1)$ *as* $n \to \infty$.

If $f(z)$ satisfies the conditions of Theorem II it can be written in the form

$$f(z) = f_1(z) + f_2(z)$$

where $f_1(z)$ has bounded coefficients, and

[8] Paley and Wiener, *Fourier Transforms in the Complex Domain*, p. 124.

$$f_2(z) = \sum_{n \in \{n_\kappa\}} a_n z^n.$$

Now $f_1(z)$ is regular in $|z| < 1$, and $f_2(z)$ is not continuable beyond its circle of convergence according to well known gap theorems. Since $f(z)$ and $f_1(z)$, and therefore $f_2(z)$, are regular in a sector of the unit circle, $f_2(z)$ must be regular in the unit circle; hence $f(z)$ is regular in $|z| < 1$.

By considering the function $f(ze^{i\theta}) - f(0)$ we suppose, without loss of generality, that $f(0) = 0$ and that $f(z)$ is bounded in a sector which includes the point $z = -1$ and is symmetric about the real axis. Let the contour Γ be the same as in **3** and let $\phi(z)$ be defined as in (3). Then $\phi(z)$ satisfies the relation (4) and so it is bounded at all integer points except possibly those of the sequence $\{n_\kappa\}$. Also

$$\phi(z) \leqq A_3 e^{k|z|}, \qquad k < \pi.$$

From Theorem III of Part I we immediately infer that $\phi(z)$ is bounded on the entire real axis, and then, as relation (4) shows, $a_n = O(1)$ as $n \to \infty$. The last statement of the theorem is proved in the same way, using Theorem III of Part I. We remark that the condition $n_{\kappa+1} - n_\kappa \to \infty$ as $n \to \infty$ in Theorem II cannot be replaced by the more general condition $n_\kappa/\kappa \to \infty$ as $\kappa \to \infty$. This is shown by the example $f(z) = \sum_1^\infty (1-z)^n z^{n!}$, which was mentioned in connection with the Pólya-Carlson theorem.

6. Finally, we remark that the condition in Theorems I and II that $f(z)$ is bounded in a sector of the unit circle is unnecessarily restrictive. It would be sufficient, for example, to suppose that $|f(re^{i\theta})| \leqq q(\theta)$ in a sector $\alpha < \theta < \beta$, $0 \leqq r < 1$, where $q(\theta)$ is some function which is Lebesgue integrable over (α, β). With this hypothesis we would again need only to consider the case in which $\alpha < \pi < \beta$. We would then choose an $\epsilon > 0$ such that $\pi - \epsilon > \alpha$, $\pi + \epsilon < \beta$, and also such that $q(\pi - \epsilon) + q(\pi + \epsilon) < \infty$. In this case the function $F(re^{i\theta})$, $F(\omega) = \int_0^\omega f(t) t^{-1} dt$, would converge almost everywhere in $(\pi - \epsilon, \pi + \epsilon)$ to an absolutely continuous function as $r \to 1$. The subsequent proof would be essentially unchanged.

We also note that it is possible to prove Theorem I in another way; namely, by combining the method of Szegö with a method of Dienes.[9]

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[9] P. Dienes, *The Taylor Series*, Oxford (1931), p. 504.

NOTE CONCERNING THE CONFORMAL AND EQUILONG GEOMETRIES OF FOURTH AND FIFTH ORDER HORN SETS.*

By Mary Elizabeth Ladue.

The existence of a unique nontrivial equilong invariant and of a unique nontrivial conformal invariant has been proved for each horn angle of a given order of contact and the actual formulae for several of these invariants have been found.[1] The theorem stated below gives, in terms of the 1943 paper, two previously unpublished formulae for the equilong measures of fourth and fifth order horn angles.

Theorem I. *A horn angle of fourth order has the expression*

$$\frac{(\Delta a_5)^5}{(\Delta a_6)^4 - 3\Delta a_5(\Delta a_6)^2\Delta a_7 + (\Delta a_5)^2[2\Delta a_6\Delta a_8 + (\Delta a_7)^2] - (\Delta a_5)^3\Delta a_9} \tag{1}$$

as its unique equilong invariant and the fifth order horn angle has

$$\frac{(\Delta a_6)^6}{\begin{array}{l}(\Delta a_7)^5 - 4\Delta a_6(\Delta a_7)^3\Delta a_8 + 3(\Delta a_6)^2[(\Delta a_7)^2\Delta a_9 + \Delta a_7(\Delta a_8)^2] \\ \qquad - 2(\Delta a_6)^3(\Delta a_7\Delta a_{10} + \Delta a_8\Delta a_9) + (\Delta a_6)^4\Delta a_{11}\end{array}} \tag{2}$$

as its unique equilong invariant.

Associated with each n-th order horn set is a space of $n+1$ dimensions, called the conformal K_{n+1}-space, in which the distance between two points is given by the conformal measure of the horn angle associated with that pair of points and the fundamental group is the set of transformations, induced by conformal transformations of curves of the horn set, which leave invariant this distance. The equations of these induced transformations for the conformal K_5-space and the conformal K_6-space are given by equations (3) and (4);

* Received April 24, 1944; Revised September 9, 1944.

[1] See the author's previous paper, "Conformal geometry of Horn angles of higher order," *American Journal of Mathematics*, vol. 65 (1943), pp. 455-476, for references, definitions of terms and description of methods used in this note.

$$
(3)\quad \begin{aligned}
V &= m_1^4 v + h_1,\\
W &= m_1^5 w + 3m_1^3 m_2 v + h_2,\\
X &= m_1^6 x + 4m_1^4 m_2 w + 2m_1^2 m_3 v + h_3,\\
Y &= m_1^7 y + 5m_1^5 m_2 x + m_1^3(3m_3 + m_2^2) w + m_1 m_4 v + h_4,\\
Z &= m_1^8 z + 6m_1^6 m_2 y + m_1^4(4m_3 + m_2^2) x\\
&\quad + 2m_1^2(m_4 - m_2 m_3 + 3m_2^3) w\\
&\quad + m_1^4(6m_2 m_4 + 4m_3^2 - 54m_2^2 m_3 + 81 m_2^4) v + h_5;
\end{aligned}
$$

$$
(4)\quad \begin{aligned}
U &= m_1^5 u + h_1,\\
V &= m_1^6 v + 4m_1^4 m_2 u + h^2,\\
W &= m_1^7 w + 5m_1^5 m_2 v + 3m_1^3 m_3 u + h_3,\\
X &= m_1^8 x + 6m_1^6 m_2 w + m_1^4(4m_3 + \tfrac{5}{3} m_2^2) v + 2m_1^2 m_4 u + h_4,\\
Y &= m_1^9 y + 7m_1^7 m_2 x + m_1^5(5m_3 + \tfrac{13}{3} m_2^2) w\\
&\quad + m_1^3(3m_4 + m_2 m_3 + \tfrac{5}{3} m_2^3) v + m_1 m_5 u + h_5,\\
Z &= m_1^{10} z + 8m_1^8 m_2 y + m_1^6(6m_3 + 8m_2^2) x\\
&\quad + m_1^4(4m_4 + 4m_2 m_3 + \tfrac{8}{3} m_2^3) w\\
&\quad + m_1^2(2m_5 - 4m_2 m_4 - 3m_3^2 + 42 m_2^2 m_3 - \tfrac{200}{3} m_2^4) v\\
&\quad + (8m_2 m_5 + 12 m_3 m_4 - 96 m_2^2 m_4\\
&\qquad - 108 m_2 m_3^2 + 768 m_2^2 m_3 - 1024 m_2^5) u + h_6.
\end{aligned}
$$

THEOREM II. *The fundamental transformation and invariant metric of the geometry of the conformal space $K_5(\gamma, \gamma', \gamma'')$ are given by equations* (1) *and* (3) *respectively; those of the geometry of the conformal space $K_6(\gamma, \gamma', \gamma'', \gamma''')$ by* (2) *and* (4).

Also associated with each n-th order horn set is another $n + 1$ dimensional space, called the equilong K'_{n+1}-space, in which the distance is given by the equilong measure and the fundamental transformation is induced by the equilong transformation of the horn set. It has been shown by Kasner and the author that the equilong K'-spaces associated with horn sets of the first, second, and third orders are identical with the corresponding conformal K-spaces of those horn sets. Computation of the equilong K'_5-space and the equilong K'_6-space shows, upon comparison with the results stated in Theorem II, that this analogy holds also for the conformal and equilong K-spaces associated with fourth and fifth order horn sets.

THEOREM III. *The geometries of the equilong and conformal K-spaces are identical for horn sets of orders one to five.*

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GEOMETRY OF SCALE CURVES IN CONFORMAL MAPS.*

By Edward Kasner and John De Cicco.

1. General summary. Let a surface Σ be mapped in any arbitrary point-to-point fashion (continuous and differentiable) upon a plane Π with cartesian coordinates (x, y). The scale function $\sigma = ds/dS$ which is the ratio of the corresponding elements of arc length in the plane Π and in the surface Σ depends, in general, not only upon the point but also upon the direction. It is independent of the direction if, and only if, the mapping of Σ upon Π is conformal.

We wish to present the fundamental theorems concerning the scale function σ in a general conformal mapping of a surface Σ upon a plane Π. A scale curve is the locus of a point along which the scale function σ does not vary. Therefore, in the conformal case, there are ∞^1 scale curves, defined by the finite equation $\sigma(x, y) = \text{const.}$ We shall be chiefly concerned with these scale curves. In non-conformal maps, which we study in other papers, σ is a function of x, y, y' and so we have ∞^2 scale curves. The application to the cartography of the sphere is given in the final section.

In our work, we shall obtain new characterizations of surfaces applicable upon surfaces of revolution. In his study of the geodesics of a surface, Kasner has obtained various properties of these surfaces applicable upon surfaces of revolution.[1] If a surface possesses an isothermal system of geodesics, then it is applicable upon a surface of revolution, the geodesics corresponding to meridians. The only surfaces upon which more than one isothermal system of geodesics can exist are the surfaces of constant gaussian curvature.[2] Kasner has also proved that the only surfaces which can be represented point by point upon a plane so that the geodesics are represented by parabolas are those of constant curvature and also special types of geodesically equivalent surfaces applicable upon certain surfaces of revolution.[3]

* Received July 1, 1944. Presented to the American Mathematical Society, February, 1944. See two related papers in *Proceedings of the National Academy of Sciences*, 1944.

[1] Kasner, "Isothermal systems of geodesics," *Transactions of the American Mathematical Society*, vol. 5 (1904), pp. 55-60.

[2] For a recent discussion of these theorems, see the paper by De Cicco, "New proofs of the theorems of Beltrami and Kasner on linear families," *Bulletin of the American Mathematical Society*, vol. 49 (1943), pp. 407-412.

[3] Kasner, "Surfaces whose geodesics may be represented in the plane by parabolas," *Transactions of the American Mathematical Society*, vol. 6 (1905), pp. 141-158.

Any family of ∞^1 curves can represent the scale curves of a conformal map of any one of a certain class of surfaces Σ upon a plane Π. If it be demanded that the gaussian curvature G be constant along the scale curves, it is found that the set of scale curves must belong to a certain special class of families of curves. That is, if $f(x, y) = \text{const.}$, represents the scale curves of a conformal map of a surface Σ upon a plane Π such that the gaussian curvature G is constant along each of the scale curves, then the function f must satisfy a certain partial differential equation of the fourth order. If our family of scale curves is defined as the integral curves of a differential equation of the first order $y' = \tan \theta(x, y)$, where θ is the inclination of the direction through the point (x, y), then the function θ must satisfy a certain partial differential equation of the third order. We shall term such a family of ∞^1 curves a family of the type λ or a *quasi-isothermal family.*

Any isothermal family of curves is of the type λ; but not every such family of curves is isothermal. Thus our new quasi-isothermal families of curves may be considered to be wide generalizations of isothermal families. For example, the family of similar ellipses $x^2 + 2y^2 = \text{const.}$, is quasi-isothermal but not isothermal.

We prove that, if an isothermal family represents the scale curves of a conformal map of a surface Σ upon a plane Π such that the gaussian curvature is constant along each of these scales, then either Σ is developable or it is a surface applicable upon a general surface of revolution. In the case of a developable surface, the scale curves can be any isothermal family; but in the case of a surface applicable upon a surface of revolution, the scale curves are either ∞^1 parallel straight lines or ∞^1 concentric circles. In the latter case, the parallels of Σ correspond to the parallel straight scales or to the concentric circular scales.

Again suppose that the quasi-isothermal family of curves is a parallel family (the orthogonal trajectories of any system of ∞^1 straight lines), then they must be parallel straight lines or concentric circles. The surface Σ is either developable or applicable upon a surface of revolution.

The only simply-infinite families of straight lines or circles that are of the type λ are the pencils of straight lines or circles. The surface Σ must again be either developable or applicable upon a surface of revolution. In the latter case, the scale curves must be either parallel straight lines or concentric circles. This is a wide generalization of theorems of Lagrange.

The application of the preceding results to any conformal map of a sphere upon a plane yields *new characterizations of the Mercator, Ptolemy (stereographic) and Lambert conical projections. These are the only conformal maps*

of a sphere upon a plane such that the scale curves form an isothermal family or a parallel family. The Mercator projection is the only conformal map of a sphere upon a plane such that the scale curves are straight lines; whereas the stereographic and Lambert conical projections are the only conformal maps of a sphere upon a plane such that the scale curves are circles. These theorems are simple but the proofs are complicated.

2. Conformal maps of a surface Σ upon a plane Π. If the mapping of Σ upon Π is conformal, the square of the linear-element dS of Σ is of the form

$$(1) \qquad dS^2 = e^{2\lambda(x,y)}(dx^2 + dy^2).$$

The scale function $\sigma = ds/dS$ is then given by $\sigma = e^{-\lambda(x,y)}$. The gaussian curvature G of the surface Σ is

$$(2) \qquad G = -e^{-2\lambda(x,y)}(\lambda_{xx} + \lambda_{yy}).$$

The geodesic curvature K of a curve upon the surface Σ is then

$$(3) \qquad K = \frac{y'' - (1 + y'^2)(\lambda_y - y'\lambda_x)}{e^{\lambda}(1 + y'^2)^{3/2}}.$$

Thus if K and k are the corresponding curvatures in the surface Σ and in the plane Π, then

$$(4) \qquad K = e^{-\lambda}\left[k - \frac{\lambda_y - y'\lambda_x}{(1 + y'^2)^{3/2}}\right].$$

3. Scale curves in conformal maps. The scale curves of the conformal map (1) are given by $\lambda(x, y) = \text{const.}$ By (3) and (4), the curvatures of the scale curves on the surface and plane are related by

$$(5) \qquad K = e^{-\lambda}[k - (\lambda_x^2 + \lambda_y^2)^{1/2}],$$

where

$$(6) \qquad k = -\frac{\lambda_y^2\lambda_{xx} - 2\lambda_x\lambda_y\lambda_{xy} + \lambda_x^2\lambda_{yy}}{(\lambda_x^2 + \lambda_y^2)^{3/2}}.$$

By (5) and (6), it is found that if the scale curves are straight lines in Π and geodesics in Σ, then λ is constant. This means that Σ is developable and the conformal map (1) is an unrolling of the surface Σ upon a plane which in turn is mapped by a similitude upon the plane Π. Henceforth we shall exclude any such map from consideration, so that λ is any non-constant function of (x, y).

Suppose now that $f(x,y)=$ const., is any simple family of curves. These can represent the scale curves of that class of surfaces for which $\lambda=\lambda(f)$. The gaussian curvature G of this class of surfaces Σ is

$$G=-e^{-2\lambda(f)}[\lambda_{ff}(f_x^2+f_y^2)+\lambda_f(f_{xx}+f_{yy})]. \tag{7}$$

4. Families of curves of the type λ (quasi-isothermal). We shall say that any simple family of curves $f(x,y)=$ const., is of the type λ or is quasi-isothermal if it represents the scale curves of a conformal map of a surface Σ upon a plane Π such that the gaussian curvature G is constant along each of the scale curves. Not any simple family of curves can satisfy these requirements. In this section, we shall derive the condition that a simple family of curves shall be of the type λ.

For this purpose, it is found convenient to introduce the linear operator

$$\mathcal{D}=f_y\partial/\partial x-f_x\partial/\partial y. \tag{8}$$

It is seen that $\mathcal{D}$ will annihilate only those functions which depend upon f alone.

Now if the gaussian curvature G is constant along each of the scale curves $f(x,y)=$ const., it follows that G must be a function of f only. Therefore using the linear operator $\mathcal{D}$ upon the equation (7), we discover that

$$\frac{\mathcal{D}(f_{xx}+f_{yy})}{\mathcal{D}(f_x^2+f_y^2)}=-\frac{\lambda_{ff}}{\lambda_f}. \tag{9}$$

Using the linear operator $\mathcal{D}$ upon this equation, we obtain the following result.

THEOREM 1. *The simple family of curves $f(x,y)=$ const. is of the type λ if and only if the function f satisfies the partial differential equation of the fourth order*

$$\mathcal{D}\,\frac{\mathcal{D}(f_{xx}+f_{yy})}{\mathcal{D}(f_x^2+f_y^2)}=0. \tag{10}$$

If $\lambda(x,y)$ obeys an equation of the form $\lambda_{xx}+\lambda_{yy}=\phi(\lambda)$, then $\lambda(x,y)=c$ is a quasi-isothermal family, and c is then called the *quasi-isothermal parameter*.

It is found that if $y'=\tan\theta(x,y)$ is the differential equation of the first order defining any family of curves of the type λ, then the inclination θ must satisfy the partial differential equation of the third order

$$\begin{aligned}(\theta_x\tan\theta-\theta_y)(\partial/\partial x+\tan\theta\,\partial/\partial y)\log(\theta_{xx}+\theta_{yy})&\\ +\tan\theta(\theta_{yy}-\theta_{xx}-4\theta_x\theta_y)+(1-\tan^2\theta)(\theta_{xy}-\theta_x^2+\theta_y^2)&=0.\end{aligned} \tag{11}$$

5. Isothermal and parallel families of the type λ. The class of quasi-isothermal families contains the isothermal families as a special subclass. According to a theorem of Lie, the simple family of curves $f(x,y) = \text{const.}$ is isothermal if and only if $\mathfrak{D}[(f_{xx}+f_{yy})/(f_x^2+f_y^2)] = 0$, or the simple family of curves defined by the differential equation of the first order $y' = \tan\theta(x,y)$ is isothermal if and only if $\theta_{xx}+\theta_{yy}=0$. Placing these conditions into (10) or (11), we find that any isothermal family is of the type λ.

THEOREM 2. *If the ∞^1 scale curves of a conformal map of a surface Σ upon a plane Π form an isothermal system such that the gaussian curvature G of Σ is constant along each of the scale curves, then Σ is either developable or applicable upon a surface of revolution. In the former case, the scale curves may be any isothermal family, but in the latter case the scale curves must be either parallel straight lines or concentric circles.*

If $f(x,y) = \text{const.}$, is an isothermal family, we may so choose the parameter that f satisfies the Laplace equation $f_{xx}+f_{yy}=0$. By (9), it then follows that

$$\lambda_{ff}\mathfrak{D}(f_x^2+f_y^2) = 0. \tag{12}$$

If $\lambda_{ff}=0$, then λ is linear in the harmonic function f only. By (1) and (2), it follows that the surface Σ is developable and $f(x,y) = \text{const.}$, can represent any isothermal family.

Consider next the condition $\lambda_{ff} \neq 0$. By the preceding equation and the condition that f is harmonic, we must have

$$\mathfrak{D}(f_x^2+f_y^2) = 0, \qquad f_{xx}+f_{yy}=0. \tag{13}$$

We have to solve these equations simultaneously.

Introduce the linear operators

$$\begin{aligned} \partial/\partial u &= \tfrac{1}{2}(\partial/\partial x - i\,\partial/\partial y), & \partial/\partial v &= \tfrac{1}{2}(\partial/\partial x + i\,\partial/\partial y), \\ \partial/\partial x &= (\partial/\partial u + \partial/\partial v), & \partial/\partial y &= i(\partial/\partial u - \partial/\partial v). \end{aligned} \tag{14}$$

Kasner has termed the operator $\partial/\partial u$ the mean derivative and the operator $\partial/\partial v$ the phase derivative. (These operators are important in the development of the geometry of polygenic functions.[4])

[4] Kasner, "The second derivative of a polygenic function," *Transactions of the American Mathematical Society*, vol. 30 (1928), pp. 803-818. Also Kasner and De Cicco. "The derivative circular congruence-representation of a polygenic function," *American Journal of Mathematics*, vol. 61 (1939), pp. 995-1003. See a paper appearing in *Scripta Mathematica*, 1945.

The equations (13) then reduce to

$$f_v^2 f_{uu} = f_u^2 f_{vv}, \qquad f_{uv} = 0, \tag{15}$$

where subscripts denote the operators (14), which behave formally like partial derivatives. These equations show that all the real solutions are equivalent by a similitude to

$$f = n \log(x^2 + y^2), \quad \text{or} \quad f = x, \tag{16}$$

where $n \neq 0$.

Therefore the surface must be applicable upon a surface of revolution. In the first case the scale curves are concentric circles and in the second case the scale curves are parallel straight lines. The corresponding linear-elements of these two types are

$$dS^2 = e^{2\lambda(x^2+y^2)}(dx^2 + dy^2), \quad \text{or} \quad dS^2 = e^{2\lambda(x)}(dx^2 + dy^2). \tag{17}$$

These are conformally equivalent by the transformation $U = \log u^n$.

(We note that in the imaginary domain, the complete solutions of (15) will also include the cases where f is monogenic in $u = x + iy$, or monogenic in $v = x - iy$. In either of these two cases, the corresponding surfaces are imaginary and are applicable to imaginary surfaces of revolution. The scale curves then are minimal lines of the same kind).

THEOREM 3. *If the ∞^1 scale curves of a conformal map of a surface Σ upon a plane Π form a parallel family such that the gaussian curvature G is constant along each of the scale curves, then the scale curves are either parallel straight lines or concentric circles, and Σ is developable or applicable upon a surface of revolution.*

The condition for a parallel family of curves is $\mathfrak{D}(f_x^2 + f_y^2) = 0$. By equation (9), it follows that $\mathfrak{D}(f_{xx} + f_{yy}) = 0$. Hence from the proof of Theorem 2, it follows that the scale curves must be parallel straight lines or concentric circles. The surface Σ must then be either developable or applicable upon a surface of revolution.

6. Simply-infinite families of straight lines which are of the type λ (quasi-isothermal). We shall discuss the following result:

THEOREM 4. *The only families of ∞ straight lines which are of the type λ are the pencils of straight lines.*

Consider the ∞^1 straight lines

$$y = x\alpha(f) + \beta(f), \tag{18}$$

where α and β are certain functions of f alone. This defines f as a certain function of (x, y). Differentiating the above partially with respect to x and y, we obtain

$$\begin{aligned} f_x^2 + f_y^2 &= \frac{1+\alpha^2}{(x\alpha_f + \beta_f)^2}, \\ f_{xx} + f_{yy} &= \frac{2\alpha\alpha_f}{(x\alpha_f + \beta_f)^2} - \frac{(1+\alpha^2)(x\alpha_{ff} + \beta_{ff})}{(x\alpha_f + \beta_f)^3}. \end{aligned} \tag{19}$$

Using the operator $\mathfrak{D}$ upon the preceding two equations, we find

$$\begin{aligned} \mathfrak{D}(f_x^2 + f_y^2) &= -\frac{2(1+\alpha^2)\alpha_f}{(x\alpha_f + \beta_f)^4}, \\ \mathfrak{D}(f_{xx} + f_{yy}) &= -\frac{4\alpha_f^2 + (1+\alpha^2)\alpha_{ff}}{(x\alpha_f + \beta_f)^4} + \frac{3(1+\alpha^2)\alpha_f(x\alpha_{ff} + \beta_{ff})}{(x\alpha_f + \beta_f)^5}. \end{aligned} \tag{20}$$

Now if $\alpha_f = 0$, our straight lines are parallel. Assuming that $\alpha_f \neq 0$, it follows from (20) that

$$\frac{\mathfrak{D}(f_{xx} + f_{yy})}{\mathfrak{D}(f_x^2 + f_y^2)} = \frac{2\alpha\alpha_f}{1+\alpha^2} + \frac{\alpha_{ff}}{2\alpha_f} - \frac{3(x\alpha_{ff} + \beta_{ff})}{2(x\alpha_f + \beta_f)}. \tag{21}$$

Applying the linear operator $\mathfrak{D}$ upon this, we find

$$\mathfrak{D}\,\frac{\mathfrak{D}(f_{xx} + f_{yy})}{\mathfrak{D}(f_x^2 + f_y^2)} = \frac{-3(\alpha_{ff}\beta_f - \alpha_f\beta_{ff})}{2(x\alpha_f + \beta_f)^3}. \tag{22}$$

By the condition (10) for quasi-isothermal systems of curves and the preceding equation, it follows that the ∞^1 straight lines must form a pencil. This completes the proof of Theorem 4.

7. Simply-infinite families of circles which are of the type λ (quasi-isothermal). Next we shall prove the following extension of the preceding result.

THEOREM 5. *The only families of ∞^1 circles which are of the type λ are the pencils of circles.*

Consider the family of ∞^1 circles

$$x^2 + y^2 = 2[xa(f) + yb(f) + c(f)], \tag{23}$$

where (a, b, c) are functions of f. This equation defines f as a function of

(x, y). We shall have occasion to use the radius r which is given by the formula

$$r^2 = a^2 + b^2 + 2c. \tag{24}$$

Differentiating (23) partially with respect to x and y, we find

$$\begin{aligned} f_x^2 + f_y^2 &= \frac{r^2}{(xa_f + yb_f + c_f)^2}, \\ f_{xx} + f_{yy} &= \frac{2rr_f}{(xa_f + yb_f + c_f)^2} - \frac{r^2(xa_{ff} + yb_{ff} + c_{ff})}{(xa_f + yb_f + c_f)^3}. \end{aligned} \tag{25}$$

Using the linear operator $\mathfrak{D}$ upon these equations, we find

$$\begin{aligned} \mathfrak{D}(f_x^2 + f_y^2) &= \frac{-2r^2[(y-b)a_f - (x-a)b_f]}{(xa_f + yb_f + c_f)^4}, \\ \mathfrak{D}(f_{xx} + f_{yy}) &= -\frac{4rr_f[(y-b)a_f - (x-a)b_f] + r^2[(y-b)a_{ff} - (x-a)b_{ff}]}{(xa_f + yb_f + c_f)^4} \\ &\quad + \frac{3r^2(xa_{ff} + yb_{ff} + c_{ff})[(y-b)a_f - (x-a)b_f]}{(xa_f + yb_f + c_f)^5}. \end{aligned} \tag{26}$$

Now if $\mathfrak{D}(f_x^2 + f_y^2) = 0$, it can be shown that the circles are concentric. Assuming that $\mathfrak{D}(f_x^2 + f_y^2) \neq 0$, it follows that

$$\begin{aligned} \frac{\mathfrak{D}(f_{xx} + f_{yy})}{\mathfrak{D}(f_x^2 + f_y^2)} &= \frac{2r_f}{r} + \frac{[(y-b)a_{ff} - (x-a)b_{ff}]}{2[(y-b)a_f - (x-a)b_f]} \\ &\quad - \frac{3(xa_{ff} + yb_{ff} + c_{ff})}{2(xa_f + yb_f + c_f)}. \end{aligned} \tag{27}$$

From this we find

$$\begin{aligned} \mathfrak{D}\frac{\mathfrak{D}(f_{xx} + f_{yy})}{\mathfrak{D}(f_x^2 + f_y^2)} &= \frac{r^2(a_{ff}b_f - a_f b_{ff})}{2[(y-b)a_f - (x-a)b_f]^2(xa_f + yb_f + c_f)} \\ &\quad - \frac{\begin{array}{l} 3[(a_{ff}b_f - a_f b_{ff})(ax + by + 2c) \\ \qquad + (b_f c_{ff} - b_{ff}c_f)(x-a) + (a_{ff}c_f - a_f c_{ff})(y-b)] \end{array}}{2(xa_f + yb_f + c_f)^3}. \end{aligned} \tag{28}$$

Upon setting the right hand side of this equation equal to zero, and using the parametric form of the ∞^1 circles: $x = a + r(1 - t^2)/(1 + t^2)$, $y = b + 2rt/(1 + t^2)$, we find

$$\begin{aligned} & r(1 + t^2)(a_{ff}b_f - a_f b_{ff})[(r_f - a_f)t^2 + 2b_f t + (r_f + a_f)]^2 \\ & - 3(b_f t^2 + 2a_f t - b_f)^2 \left[\begin{array}{l} (a_{ff}b_f - a_f b_{ff})\{(r-a)t^2 + 2bt + (r+a)\} \\ \quad + (b_f c_{ff} - b_{ff}c_f)(1 - t^2) + 2t(a_{ff}c_f - a_f c_{ff}) \end{array}\right] = 0 \end{aligned} \tag{29}$$

It is obvious that if $a_{ff}b_f - a_f b_{ff} = 0$, then this identity in t yields the pencils of circles. We shall prove now that $a_{ff}b_f - a_f b_{ff}$ must be zero. For otherwise if $a_{ff}b_f - a_f b_{ff} \neq 0$, then $a_f \neq 0$. Thus we may take $f = a$ and $b_{aa} \neq 0$. The preceding equation reduces to

$$(30)\quad rb_{aa}(1+t^2)[(r_a-1)t^2+2b_a t+(r_a+1)]^2 \\ + 3(b_a t^2+2t-b_a)^2\begin{bmatrix} -b_{aa}\{(r-a)t^2+2bt+(r+a)\} \\ +(b_a c_{aa}-b_{aa}c_a)(1-t^2)-2tc_{aa}\end{bmatrix} = 0.$$

Setting the coefficient of t^6 equal to zero, we find

$$(31)\qquad rb_{aa}(r_a-1)^2 = 3b_a^2[b_{aa}(r-a)+(b_a c_{aa}-b_{aa}c_a)].$$

By means of this equation, the equation (30) may be written as

$$(32)\quad rb_{aa}(1+t^2)[(r_a-1)t^2+2b_a t+(r_a+1)]^2 \\ = (b_a t^2+2t-b_a)[(rb_{aa}/b_a^2)(r_a-1)^2(t^2-1) \\ + 6t(bb_{aa}+c_{aa})+6rb_{aa}].$$

Upon setting the coefficient of t^5 equal to zero, we find

$$(33)\quad 6(bb_{aa}+c_{aa}) = \frac{4rb_{aa}}{b_a}(r_a-1) - \frac{4rb_{aa}}{b_a^3}(r_a-1)^2.$$

Substituting this into (32) and simplifying, we obtain

$$(34)\quad (1+t^2)[(r_a-1)t^2+2b_a t+(r_a+1)]^2 \\ = (b_a t^2+2t-b_a)^2[(1/b_a^2)(r_a-1)^2(t^2-1) \\ + (4t/b_a^3)(r_a-1)\{b_a^2-(r_a-1)\}+6].$$

Upon setting the coefficient of t equal to zero, we discover that $b_a = 0$. This is impossible. Hence the only families of ∞^1 circles which are of the type λ are the pencils of circles. This completes the proof of Theorem 5.

By Theorems 4 and 5, it follows that if the ∞^1 scale curves of a conformal map of a surface Σ upon a plane Π are straight lines or circles such that the gaussian curvature G is constant along each of the scale curves, then Σ is developable or applicable upon a surface of revolution. In the latter case, the straight scales must be parallel, or the circular scales must be concentric.

8. Application of the preceding results to the sphere. The two important types of conformal maps (17) upon a plane Π of a surface Σ applicable upon a surface of revolution are reducible under the conformal transformation $U = \log u^n$ to the single case

(35) $$dS^2 = e^{2\lambda(x)}(dx^2 + dy^2).$$

The gaussian curvature G is then

(36) $$G = -e^{-2\lambda(x)}\lambda_{xx}.$$

Let us now suppose that the surface Σ is a sphere so that the gaussian curvature G is a positive constant. By the application of an appropriate magnification, G may be reduced to unity, that is, $G = 1$. The complete integration of (36) where $G = 1$ shows that by applying a proper similitude, the linear-element (35) of the sphere Σ may be written as

(37) $$dS^2 = \operatorname{sech}^2 x\,(dx^2 + dy^2).$$

By applying the transformation $U = \log u^n$ to this, we find that the other important conformal map (17) of a sphere Σ upon a plane Π may be written as

(38) $$dS^2 = \frac{4n^2(x^2+y^2)^{n-1}(dx^2+dy^2)}{[(x^2+y^2)^n+1]^2}.$$

The first map (37) represents a Mercator projection and the second map (38) is a stereographic (Ptolemy) projection for $n = 1$, and a Lambert conical projection for $n \neq 1$.

Thus if a sphere is mapped conformally upon a plane such that its scale curves form an isothermal family, then the conformal map must be either a Mercator, or a Ptolemy stereographic projection, or a Lambert projection. Also these are the only conformal maps of a sphere upon a plane such that the scale curves form a parallel family. A Mercator projection is the only conformal map of a sphere upon a plane such that its scale curves are straight lines. Finally stereographic projections and Lambert conical projection are the only conformal maps of a sphere upon a plane such that their scale curves are circles.

It is impossible to map a sphere conformally on a plane so that all the ∞^2 great circles become straight. But it can be shown from our work, that the three classic maps (Ptolemy, Mercator, Lambert) are the only conformalities which map the maximum possible number of geodesics (namely ∞^1) into straight lines.

Non-conformal maps of a general surface and of a sphere will be studied in our next paper. The scale curves then form a two-parameter family, so the theory is radically different. See *Science*, vol. 98 (1943), pp. 324-325.

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GIBBS' PHENOMENON AND THE PRIME NUMBER THEOREM.*

By Aurel Wintner.

If $\epsilon > 0$ and $T > \epsilon$ are fixed, the series

(1) $$\sum_p p^{-1} \sin (t \log p),$$

being the imaginary part of

$$\sum_p p^{-1-it},$$

is uniformly convergent for $\epsilon < t < T$ (and therefore for $-T < t < -\epsilon$); in particular, (1) is convergent for every t. According to Mertens [2], these facts are substantially equivalent to the non-vanishing of $\zeta(s)$ on the line $\sigma = 1$, and therefore to the prime theorem (Ikehara). It is understood that p runs *increasingly* through all primes.

Since $\sum_p p^{-1} = \infty$, the series

(2) $$\sum_p p^{-1} \cos (t \log p)$$

cannot be uniformly convergent near $t = 0$ (it is uniformly convergent for $\epsilon < t < T$, since it is the real part of $\sum p^{-1-it}$). Due to the saltus of $\arg \zeta(1 + it)$ at $t = 0$, where $\zeta(s) \sim (s-1)^{-1}$ and $s = 1 + it$, *the series* (1) *is not uniformly convergent near* $t = 0$.

This suggests a certain analogy between (2), (1) and the classical Fourier series

$$\sum_{n=1}^{\infty} n^{-1} \cos nt, \qquad \sum_{n=1}^{\infty} n^{-1} \sin nt.$$

In fact, the latter Fourier series belong to the Maclaurin series of the logarithm of $(1-z)^{-1}$ in the same way as (2), (1) belong (except for trivial terms) to the Dirichlet series of the logarithm of $\zeta(s) \sim (s-1)^{-1}$, where $z = e^{it}$ and $s = 1 + it$ respectively. Moreover, both series (2), (1) are Fourier series in the sense (B^2) and belong, in this sense, precisely to the functions to which they converge (if $t \neq 0$); cf. Wintner [4].

In view of this parallelism, one will expect that, corresponding to the classical Gibbs phenomenon of the series

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$$\sum_{n=1}^{\infty} n^{-1} \sin nt,$$

the partial sums of the series (1) *are uniformly bounded near* $t = 0$.

More specifically, it will be shown that *the partial sums of the series* (1) *exhibit a Gibbs phenomenon at* $t = 0$; *in the sense that, as* $m \to \infty$, *the difference*

$$\sum_{p \leqq m} p^{-1} \sin (t \log p) - S(t \log m) \tag{3}$$

tends to a limit uniformly for $|t| < T$, *where* T *is arbitrarily fixed and*

$$S(x) = \int_0^x u^{-1} \sin u \, du \tag{4}$$

(so that $S(\pm \infty) = \pm \frac{1}{2}\pi$).

The proof is suggested by the formal remark that the ordinary Dirichlet series

$$-\sum_{n=2}^{\infty} (\log n)^{-1} n^{-s} \tag{5}$$

and the (aperiodic) trigonometric series (1) are substantially the integrals of

$$\sum_{n=1}^{\infty} n^{-s} = \zeta(s) \sim (s-1)^{-1}, \qquad (s \to 1), \tag{6}$$

and of the imaginary part of $\log \zeta (1 + it)$, respectively. Correspondingly, (1) will first be replaced by

$$\sum_{n=2}^{\infty} (n \log n)^{-1} \sin (t \log n), \tag{7}$$

that is, by the imaginary part of (5) on the line $\sigma = 1$, where $s = \sigma + it$. It will then be shown that the uniformity statement made with regard to (3) remains unaffected if the partial sums of (1) occurring in (3) are replaced by the corresponding partial sums of (7).

It is a standard fact, readily verified by partial summation, that the series

$$\sum_{n=1}^{\infty} n^{-1-it}$$

of $\zeta(1 + it)$ is divergent for every t and that the partial sums of this series are uniformly bounded for $\epsilon < t < T$, if $0 < \epsilon < T < \infty$ (but not if $\epsilon = 0$). Since $(\log n)^{-1}$ tends decreasingly to 0, it follows from Dedekind's convergence criterion (that is, again by partial summation) that the series (5),

where $s = 1 + it$, and therefore also the imaginary part, (7), of this series, is uniformly convergent for $\epsilon < t < T$.

In order to control the partial sums of (7) as $\epsilon \to 0$, use will be made of an observation made in 1862 by Kinkelin, according to which

$$\zeta(1 + it) = \sum_{n=1}^{m} n^{-1-it} + (it\, m^{it})^{-1} + o(1) \quad \text{as} \quad m \to \infty$$

holds *uniformly* on every *fixed* open interval $\epsilon < t < T$, as seen by partial summation (deeper results, valid *within* the critical strip, are contained in the approximate functional equation of Riemann and Hardy-Littlewood; cf. [1]). Hence, if $\zeta^*(s)$ denotes the entire function

$$\zeta^*(s) = \zeta(s) - 1 - (s-1)^{-1},$$

it follows, on writing u instead of t, that

$$\zeta^*(1 + iu) - iu^{-1} = \sum_{n=2}^{m} n^{-1-iu} - im^{-iu}u^{-1} + o(1) \quad \text{as} \quad m \to \infty$$

holds uniformly for $0 < u < T$, if T is fixed. For the real parts, this implies that

$$\Re \sum_{n=2}^{m} n^{-1-iu} = u^{-1} \sin(u \log m) + \Re\zeta^*(1 + iu) + o(1) \quad \text{as} \quad m \to \infty,$$

where the o-term is uniform for $0 < u < T$. Consequently, as $m \to \infty$,

$$\int_0^t \Re \sum_{n=2}^{m} n^{-1-iu} du - \int_0^t u^{-1} \sin(u \log m)\, du \to \int_0^t \Re\zeta^*(1 + iu)\, du$$

holds uniformly for $0 < u < T$. But the second integral on the left is $S(t \log m)$, by (4); while the first integral on the left is the m-th partial sum of the series (7), since, in accordance with the remark made in connection with (5),

$$-i \int_0^t \sum_{n=2}^{m} n^{-1-iu}\, du = \sum_{n=2}^{m} (n \log n)^{-1}(n^{-it} - 1).$$

Accordingly, as $m \to \infty$,

$$(8) \qquad \sum_{n=2}^{m} (n \log n)^{-1} \sin(t \log n) - S(t \log m) \to \int_0^t \Re\zeta^*(1 + iu)\, du$$

holds uniformly for $0 < t < T$, where T is arbitrarily fixed.

It follows that, in order to prove that the expression (3) tends to a limit uniformly for $0 < t < T$, it is sufficient to show that, as $m \to \infty$, the difference between the first term, Σ, of the expression (3) and the m-th partial sum of the series (7) tends to 0 uniformly for $0 < t < T$. In other words, it is sufficient to prove that the series

$$\sum_{k=1}^{\infty} a_k \sin(t \log k) \tag{9}$$

is uniformly convergent for $0 < t < T$, where the coefficients a_k are defined by

$$\sum_{k=1}^{\infty} a_k k^{-s} = \sum_p p^{-s} - \sum_{n=2}^{\infty} (\log n)^{-1} n^{-s}, \qquad (\sigma > 1). \tag{10}$$

But if $s = 1 + it$, then (9) is the negative of the imaginary part of (10). Consequently, it is sufficient to show that the Dirichlet series (10) is uniformly convergent on every fixed segment $|t| < T$ of the line $\sigma = 1$. And this can be proved as follows:

Since $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$ for $\sigma > 1$,

$$\sum_p p^{-s} = \log \zeta(s) + g(s),$$

where $g(s)$ is a function regular for $\sigma > \frac{1}{2}$. On the other hand, since $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $\sigma > 1$,

$$\sum_{n=2}^{\infty} (\log n)^{-1} n^{-s} = \int_2^s \zeta(z)\, dz - s + \text{const.}$$

for $\sigma > 1$. Hence, from (10),

$$\sum_{k=1}^{\infty} a_k k^{-s} = \log \zeta(s) + \int_2^s \zeta(z) dz + f(s), \qquad (\sigma > 1), \tag{11}$$

where $f(s)$ is a function regular for $\sigma > \frac{1}{2}$. But

$$\zeta(s) = (s-1)^{-1} + \text{an entire function} \tag{12}$$

and, by the function-theoretical formulation of the prime number theorem,

$$\zeta(s) \neq 0 \quad \text{for} \quad \sigma \geqq 1. \tag{13}$$

Since (12) and (13) obviously imply that the sum of the first two terms on the right of (11) is regular for $\sigma \geqq 1$, and since the third term, $f(s)$,

is regular for $\sigma > \frac{1}{2}$ and therefore for $\sigma \geqq 1$, it follows from (11) that the difference on the right of (10) is regular for $\sigma \geqq 1$.

This is the relevant consequence of the prime number theorem. In fact, the balance of the proof will in no sense involve the prime number theorem.

According to the analogue of the Tauberian theorem of P. Fatou–M. Riesz for Dirichlet series

$$\sum_{k=1}^{\infty} a_k k^{-s},$$

(cf. M. Riesz [3]), such a series must converge uniformly on every fixed segment $-T < t < T$ of the line $\sigma = 1$, if, on the one hand,

$$\sum_{k=1}^{x} a_k = o(x) \quad \text{as} \quad x \to \infty \tag{14}$$

and, on the other hand, the Dirichlet series converges in the open half-plane $\sigma > 1$ to a function which remains regular on the line $\sigma = 1$. Since the latter assumption has just been assured for (10), all that remains to be proved is that (14) is satisfied when a_k is defined by (10).

It is clear from (10) that the sum (14) is now identical with

$$\sum_{k=1}^{x} a_k = \sum_{p \leqq x} 1 - \sum_{n=2}^{x} (\log n)^{-1}. \tag{15}$$

But the first term on the right of (15) is the number of primes not exceeding x, and therefore, according to Chebyshev, it is majorized by a constant multiple of $x/\log x$, which is $o(x)$. Since the second term on the right of (15) is

$$\sum_{n=2}^{x} (\log n)^{-1} = \sum_{n=2}^{x} o(1) = o(x),$$

the proof is complete.

Since only the roughest estimates were needed in this verification of (14), one might have the impression that the application of a Tauberian theorem can be avoided, if the formulation $\pi(x) \sim x/\log x$ of the prime number theorem is used in (15). In fact, if (15) is written in the form

$$\sum_{k=1}^{x} a_k = \pi(x) - \sum_{n=2}^{x} (\log n)^{-1} \sim \pi(x) - \int_{2}^{x} (\log u)^{-1} du \sim \pi(x) - x/\log x,$$

then $\pi(x) \sim x/\log x$ improves the $o(x)$ in (14) to $o(x/\log x)$. However, this does not suffice for the elimination of the Tauberian theorem. What would suffice is the improvement of the estimate $o(x/\log x)$ to $o(x/\log^{1+\epsilon} x)$ for some $\epsilon > 0$, since the proof could then be based on

$$\int_2^\infty |o(x/\log^{1+\epsilon} x)|\,dx < \infty, \tag{17}$$

where $\epsilon > 0$ could be arbitrarily small. But the fact is that, while even de la Vallée–Poussin's zero-free domain implies much more than the truth of $\pi(x) - x/\log x = o(x/\log^\beta x)$ for some $\beta > 1$, the prime number theorem itself, which is precisely (13), supplies only $\beta = 1$, and therefore no $\beta = 1 + \epsilon > 1$ satisfying (17).

This methodical situation is pointed out because the above proof is such as to avoid the prime number theorem entirely, if only the question of Gibbs' phenomenon, i. e., the vicinity of the point $t = 0$, is concerned. In fact, it then suffices to apply the Tauberian theorem for a *fixed* (small) T.

On the other hand, if T is allowed to be arbitrarily large, then the above procedure is reversible, i. e., *the uniform convergence of the Dirichlet series* (10) *on every fixed segment* $-T < t < T$ *of the line* $\sigma = 1$ *is equivalent to the prime number theorem,*

$$p_n \sim n \log n. \tag{18}$$

In fact, if the Dirichlet series (10) converges on the line $\sigma = 1$ to a continuous function, then it is clear from (11) that (12) implies (13).

In this connection, it is worth mentioning that *Riemann's hypothesis is equivalent to the convergence of the Dirichlet series* (10) *for* $\sigma > \frac{1}{2}$. In fact, Riemann's hypothesis is equivalent to the truth of

$$\pi(x) - \mathrm{Li}\,(x) = o(x^{\frac{1}{2}+\epsilon}), \quad \text{where} \quad \mathrm{Li}\,(x) \equiv \int_2^x (\log u)^{-1}\,du, \tag{19}$$

for every $\epsilon > 0$. But (19) is just a rewording of the statement that the abscissa of convergence of the Dirichlet series (10) does not exceed $\frac{1}{2} + \epsilon$.

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